ON THE PROBABILITY OF THE OCCURRENCE OF AT LEAST m EVENTS AMONG n ARBITRARY EVENTS

By Kai Lai Chung

Tsing Hua University, Kunming, China

Introduction. Let $E_1, \cdots, E_n$, denote n arbitrary events. Let $p_{\nu'_1 \cdots \nu'_i \nu_{i+1} \cdots \nu_j}$, where $0 \leqq i \leqq j \leqq n$ and $(\nu_1, \cdots, \nu_j)$ is a combination of the integers $(1, \cdots, n)$, denote the probability of the non-occurrence of $E_{\nu_1}, \cdots, E_{\nu_i}$ and the occurrence of $E_{\nu_{i+1}}, \cdots, E_{\nu_j}$. Let $p_{[\nu_1 \cdots \nu_i]}$ denote the probability of the occurrence of $E_{\nu_1}, \cdots, E_{\nu_i}$ and no others among the n events. Let $S_j = \Sigma p_{\nu_1 \cdots \nu_j}$ where the summation extends to all combinations of j of the n integers $(1, \cdots, n)$. Let $p_m(\nu_1, \cdots, \nu_k)$, $(1 \leqq m \leqq k \leqq n)$, denote the probability of the occurrence of at least m events among the k events $E_{\nu_1}, \cdots, E_{\nu_k}$.

By the set $(x_1, \cdots, x_b, \cdots, x_a) - (x_1, \cdots, x_b)$ (where $b \leqq a$) we mean the set $(x_{b+1}, \cdots, x_a)$. And by a $\binom{a}{b}$-combination out of $(x_1, \cdots, x_a)$ we mean a combination of b integers out of the a integers $(x_1, \cdots, x_a)$.

We often use summation signs with their meaning understood, thus for a fixed k, $1 \leqq k \leqq n$, the summations in $\Sigma p_{\nu_1 \cdots \nu_k}$, or $\Sigma p_m(\nu_1, \cdots, \nu_k)$, extend to all the $\binom{n}{k}$-combinations out of $(1, \cdots, n)$.

The following conventions concerning the binomial coefficients are made:

$$\binom{0}{0} = 1, \qquad \binom{a}{b} = 0 \qquad \text{if} \qquad a < b \qquad \text{or if} \qquad b < 0.$$

It is a fundamental theorem in the theory of probability that, if $E_1, \cdots, E_n$ are incompatible (or "mutually exclusive"), then

$$p_1(1, \cdots, n) = p_1 + \cdots + p_n .$$

When the events are arbitrary, we have Boole's inequality

$$p_1(1, \cdots, n) \leqq p_1 + \cdots + p_n .$$

Gumbel[1] has generalized this inequality to the following:

$$p_1(1, \cdots, n) \leqq \frac{\Sigma p_1(\nu_1, \cdots, \nu_k)}{\binom{n-1}{k-1}},$$

[1] *C. R. Acad. Sc.* Vol. 205(1937), p. 774.

for $k = 1, \cdots, n$. The case $k = 1$ gives Boole's inequality. Fréchet[2] has announced that Gumbel's result can be sharpened to the following

$$(1) \qquad A_{k+1} = \frac{\Sigma p_1(\nu_1, \cdots, \nu_{k+1})}{\binom{n-1}{k}} \leqq \frac{\Sigma p_1(\nu_1, \cdots, \nu_k)}{\binom{n-1}{k-1}} = A_k,$$

for $k = 1, \cdots, n-1$. Thus, A_k is non-increasing for k increasing. On the other hand, Poincaré has obtained the following formula which expresses $p_1(1, \cdots, n)$ in terms of the S_j's,

$$(2) \qquad \begin{aligned} p_1(1, \cdots, n) = \sum p_{\nu_1} - \sum p_{\nu_1\nu_2} + \sum p_{\nu_1\nu_2\nu_3} - \cdots \\ + (-1)^n p_{1\cdots n} = \sum_{j=1}^{n} (-1)^{j-1} S_j. \end{aligned}$$

In the present paper we shall study the more general function $p_m(\nu_1, \cdots, \nu_k)$ as defined above. First we generalize Poincaré's formula and Fréchet's inequalities. In Theorem 1 we establish (for $1 \leqq m \leqq n$)

$$(3) \qquad \begin{aligned} p_m(1, \cdots, n) &= \sum p_{\nu_1\cdots\nu_m} - \binom{m}{1} \sum p_{\nu_1\cdots\nu_{m+1}} \\ &\quad + \binom{m+1}{2} \sum p_{\nu_1\cdots\nu_{m+2}} + \cdots + (-1)^{n-m}\binom{n-1}{m-1} p_{1\cdots n} \\ &= \sum_{i=0}^{n-m} (-1)^i \binom{m+i-1}{i} S_{m+i}. \end{aligned}$$

Although this result is well known, we prove it in preparation for Theorem 2. Theorem 3 establishes

$$(4) \qquad A_{k+1}^{(m)} = \frac{\Sigma p_m(\nu_1, \cdots, \nu_{k+1})}{\binom{n-m}{k+1-m}} \leqq \frac{\Sigma p_m(\nu_1, \cdots, \nu_k)}{\binom{n-m}{k-m}} = A_k^{(m)},$$

for $k = 1, \cdots, n-1$ and $1 \leqq m \leqq k$.

Next, we extend the inequalities (4), and in Theorem 4 we show that

$$(5) \qquad A_k^{(m)} \leqq \tfrac{1}{2}(A_{k-1}^{(m)} + A_{k+1}^{(m)});$$

which states that the differences $A_k - A_{k+1}$ $(k = 1, \cdots, n-1)$ are non-decreasing for increasing k. From this and a simple result we can deduce (4). Also Theorem 2 establishes that

$$(6) \qquad \sum_{i=0}^{2l+1} (-1)^i \binom{m+i-1}{i} S_{m+i} \leqq p_m(1, \cdots, n) \leqq \sum_{i=0}^{2l} (-1)^i \binom{m+i-1}{i} S_{m+i},$$

[2] Loc. cit., Vol. 208(1939), p. 1703.

for $2l + 1 \leqq n - m$ and $2l \leqq n - m$ respectively. These inequalities throw light on formula (3) and are sharper than the following analogue of Boole's inequality for $p_m(1, \cdots, n)$, which is a special case of (4):

$$p_m(1, \cdots, n) \leqq \Sigma p_{\nu_1 \cdots \nu_m}. \tag{7}$$

The last statement will be evident in the proof.

In Theorem 5 we give an "inversion" of the formula (3), i.e. we express $p_{1 \cdots n}$ in terms of the $p_m(\nu_1, \cdots, \nu_k)$'s, as follows:

$$\begin{aligned}\binom{n-1}{m-1} p_{1 \cdots n} &= \sum p_m(\nu_1, \cdots, \nu_m) - \sum p_m(\nu_1, \cdots, \nu_{m+1}) + \cdots \\ &\qquad + (-1)^{n-m} p_m(1, \cdots, n) \\ &= \sum_{i=0}^{n-m} (-1)^i \sum p_m(\nu_1, \cdots, \nu_{m+i}).\end{aligned} \tag{8}$$

This of course implies the following more general formula for $p_{\alpha_1 \cdots \alpha_r}$,

$$\binom{r-1}{m-1} p_{\alpha_1 \cdots \alpha_r} = \sum_{i=0}^{r-m} (-1)^i \sum p_m(\nu_1, \cdots, \nu_{m+i})$$

where $(\alpha_1, \cdots, \alpha_r)$ is a combination of the integers $(1, \cdots, n)$ and where the second summation extends to all the $\binom{r}{m+i}$-combinations of $(\alpha_1, \cdots, \alpha_r)$. Since it is known[3] that we can express other functions such as S_r, $p_{[\mu_1 \cdots \mu_r]}$ in terms of the $p_{\mu_1 \cdots \mu_r}$'s, we can also express them in terms of the $p_m(\nu_1, \cdots, \nu_k)$'s, provided $r \geqq m$.

Finally, for the case $m = 1$, we give in Theorem 6 an explicit formula for $p_{[1 \cdots r]}$ in terms of the $p_1(\nu_1, \cdots, \nu_k)$'s, as shown in (9),

$$\begin{aligned}p_{[1 \cdots r]} &= -p_1(r+1, \cdots, n) + \sum_{\nu_1} p_1(\nu_1, r+1, \cdots, n) \\ &\qquad - \sum_{\nu_1, \nu_2} p_1(\nu_1, \nu_2, r+1, \cdots, n) + \cdots \\ &\qquad + (-1)^{r-1} \sum p_1(1, \cdots, r, r+1, \cdots, n), \\ &= \sum_{i=1}^{r} (-1)^{i-1} \sum_{(\nu_1, \cdots, \nu_i)} p_1(\nu_1, \cdots, \nu_i, r+1, \cdots, n),\end{aligned} \tag{9}$$

where $(\nu_1, \cdots, \nu_i)$ runs through all the $\binom{r}{i}$-combinations from $(1, \cdots, r)$. This of course implies the following more general formula:

$$p_{[\alpha_1 \cdots \alpha_r]} = \sum_{i=1}^{r} (-1)^{i-1} \sum_{(\nu_1, \cdots, \nu_i)} p_1(\nu_1, \cdots, \nu_i, \alpha_{r+1}, \cdots, \alpha_n),$$

[3,4] Fréchet, "Condition d'existence de systemes d'événements associés à certaines probabilités," *Jour. de Math.*, (1940), p. 51–62.

where $(\alpha_1, \cdots, \alpha_r, \cdots \alpha_n)$ is a permutation of $(1, \cdots, n)$ and where $(\nu_1, \cdots, \nu_i)$ runs through all the $\binom{r}{i}$-combinations out of $(\alpha_1, \cdots, \alpha_r)$. From Theorem 6 and two lemmas we deduce a condition of existence of systems of events associated with the probabilities $p_1(\nu_1, \cdots, \nu_m)$. The author has not been able to obtain similar elegant results for the general m. Probably they do not exist.

2. Generalization of Poincaré's formula; Generalization and sharpening of Boole's inequality.

THEOREM 1:

$$p_m(1, \cdots, n) = \sum p_{\nu_1 \cdots \nu_m} - \binom{m}{1} \sum p_{\nu_1 \cdots \nu_{m+1}} + \binom{m+1}{2} \sum p_{\nu_1 \cdots \nu_{m+2}} - \cdots + (-1)^{n-m} \binom{n-1}{n-m} p_{1 \cdots n}. \tag{3}$$

PROOF: We have

$$p_m(1, \cdots, n) = \sum_{b=0}^{n-m} \sum p_{[\mu_1 \cdots \mu_{m+b}]}, \tag{10}$$

where the second summation extends, for a fixed b, to all the $\binom{n}{m+b}$-combinations of $(1, \cdots, n)$. Further we have

$$p_{\nu_1 \cdots \nu_{m+c}} = \sum_{d=0}^{n-m-c} \sum p_{[\nu_1 \cdots \nu_{m+c} \cdots \nu_{m+c+d}]} \tag{11}$$

where the second summation extends, for a fixed d, to all the $\binom{n-m-c}{d}$-combinations of $(1, \cdots, n) - (\nu_1, \cdots, \nu_{m+c})$. The formulas (10) and (11) are evident by observing that the probabilities in the summations are all additive. Now we count the number of times a fixed $p_{[\mu_1 \cdots \mu_{m+b}]}$ appears in (3). By (11) this is equal to the sum

$$\binom{m+b}{m} - \binom{m}{1}\binom{m+b}{m+1} + \binom{m+1}{2}\binom{m+b}{m+2} - \cdots + (-1)^{n-m}\binom{n-1}{n-m}\binom{m+b}{m+b} = 1,$$

since this number is the coefficient of $(-1)^m x^m$ in the expansion of

$$(1-x)^{m+b}\left(1-\frac{1}{x}\right)^{-m} = (-1)^{-m} x^m (1-x)^b.$$

Thus by (10) we have (3).

THEOREM 2: *For $2l \leqq n - m$ and $2l \leqq n - m$ respectively, we have*

$$(6)\quad \sum_{i=0}^{2l+1} (-1)^i \binom{m+i-1}{i} S_{m+i} \leqq p_m(1, \cdots, n) \leqq \sum_{i=0}^{2l} (-1)^i \binom{m+i-1}{i} S_{m+i}.$$

PROOF: By the reasoning in the previous proof, it is sufficient (in fact also necessary) to show that

$$\sum_{i=0}^{2l} \binom{m-1+i}{i}\binom{m+b}{m+i} \geqq 1, \qquad \sum_{i=0}^{2l+1} \binom{m-1+i}{i}\binom{m+b}{m+i} < 1.$$

Since

$$\binom{m-1+i}{i}\binom{m+b}{m+i} = \frac{(m+b)!}{(m-1)!\,b!}\binom{b}{i}\frac{1}{m+i}$$

is an integer, it is sufficient to show that

$$(12)\qquad \sum_{i=0}^{2l} (-1)^i \binom{b}{i}\frac{1}{m+i} > 0, \qquad \sum_{i=0}^{2l+1} (-1)^i \binom{b}{i}\frac{1}{m+i} \leqq 0.$$

Suppose $b > 0$ is even. For $i \leqq b/2 - 1$, we have $\frac{b-i}{i+1} > 1$ so that $\frac{b-i}{i+1} \geqq \frac{i+2}{i+1}$. Also $\frac{m+i}{m+i+1} \geqq \frac{i+1}{i+2}$ for $m \geqq 1$. Hence

$$\binom{b}{i+1}\frac{1}{m+i+1} = \frac{b-i}{i+1}\,\frac{m+i}{m+i+1}\binom{b}{i}\frac{1}{m+i}$$
$$\geqq \frac{i+2}{i+1}\,\frac{i+1}{i+2}\binom{b}{i}\frac{1}{m+i} = \binom{b}{i}\frac{1}{m+i}.$$

For $i \geqq b/2$ we have $\frac{b-i}{i+1} < 1$ so that $\frac{b-i}{i+1}\,\frac{m+i}{m+i+1} < 1$ and

$$\binom{b}{i+1}\frac{1}{m+i+1} < \binom{b}{i}\frac{1}{m+i}.$$

Thus the absolute values of the terms of the alternating series

$$\sum_{i=0}^{b} (-1)^i \binom{b}{i}\frac{1}{m+i} = \frac{b!}{(m+b)!(m-1)!}$$

are monotone increasing as long as $i \leqq \frac{b}{2} - 1$, reaching maximum at $i = \frac{b}{2}$ and then become monotone decreasing.

Therefore (12) evidently holds for $2l \leqq b/2$ and $2l + 1 \leqq b/2$ respectively. For $t \geqq \frac{b}{2} + 1$ we write

$$\sum_{i=0}^{t} (-1)^i \binom{b}{i}\frac{1}{m+i} = \frac{b!}{(m+b)!(m-1)!} - \sum_{i=t+1}^{b} (-1)^i \binom{b}{i}\frac{1}{m+i}$$
$$= \frac{b!}{(m+b)!(m-1)!} - \sum_{j=0}^{b-t-1} (-1)^j \binom{b}{j}\frac{1}{m+b-j}.$$

From the above and the fact that $\frac{b!}{(m+b)!(m-1)!} \leqq \frac{1}{m+b}$ we see that the righthand side is an alternating series whose terms are non-decreasing in absolute values. Hence (12) is true.

If b is odd, the case is similar.

3. Generalization of Fréchet's inequalities and related inequalities. Before proving our remaining theorems, we shall give a more detailed account of the general method which will be used. In the foregoing work we have already given two different expressions for the function $p_m(1, \cdots, n)$, namely, formulas (3) and (10), but they are not convenient for our later purposes. Formula (3) is inconvenient because it is not additive and because the $p_{\nu_1 \cdots \nu_i}$'s are related in magnitudes; while formula (10) has gone so far in the separation of the additive constituents that its application raises algebraical difficulties. Let us therefore take an intermediate course.

Let each $\binom{n}{m}$-combination $(\nu_1, \cdots, \nu_m)$ out of $(1, \cdots, n)$ be written so that $\nu_1 < \nu_2 < \cdots < \nu_m$. Then we arrange them in an ordered sequence in the following way: the combination $(\nu_1, \cdots, \nu_m)$ is to precede the combination $(\mu_1, \cdots, \mu_m)$ if, for the first $\nu_i \neq \mu_i$, we have $\nu_i > \mu_i$. After such an arrangement we symbolically denote these combinations by

$$\text{I, II}, \cdots, \left[\binom{n}{m}\right].$$

Further, all the $\binom{k}{m}$-combinations out of $(\nu_1, \cdots, \nu_k)$ where the latter is a combination out of $(1, \cdots, n)$ are arranged in the order in which they appear in the sequence just written. For example, all the $\binom{4}{2}$-combinations out of $(1, 2, 3, 4)$ are ordered thus:

$$(12) \quad (13) \quad (14) \quad (23) \quad (24) \quad (34).$$

Let U denote a typical combination $(\mu_1, \cdots, \mu_m)$. By E_U we mean the combination of events $E_{\mu_1}, \cdots, E_{\mu_m}$ so that $p_U = p_{\mu_1 \cdots \mu_m}$. In general, let the combinations $U_1, \cdots, U_{b-1}, U_b$ be given, then $p_{U_1' \cdots U_{b-1}' U_b}$ denotes the probability of the non-occurrence of $U_1, \cdots, U_{b-1}$ and the occurrence of U_b.

Now let I, II, $\cdots$, $\left[\binom{k}{m} - 1\right] = Y$, $\left[\binom{k}{m}\right] = Z$ denote all the $\binom{k}{m}$-combinations out of $(\nu_1, \cdots, \nu_k)$ in their assigned order. We have

$$p_m(\nu_1, \cdots, \nu_k) = p_{I} + p_{I'II} + p_{I'II'III} + \cdots + p_{I' \cdots Y'Z}. \tag{13}$$

This fundamental formula is evident. Of course it is possible to identify the p's on the right-hand side with the ordinary $p_{\nu_1' \cdots \nu_j}$'s, but we shall refrain from so doing and be content with the following example:

$$p_2(1, 2, 3, 4) = p_{12} + p_{12'3} + p_{12'3'4} + p_{1'23} + p_{1'23'4} + p_{1'2'34}.$$

THEOREM 3. *For $k = 1, \cdots, n-1$ and $1 \leqq m \leqq k$ we have*

$$\binom{n-m}{k-m} \Sigma p_m(\nu_1, \cdots, \nu_{k+1}) \leqq \binom{n-m}{k+1-m} \Sigma p_m(\nu_1, \cdots, \nu_k).$$

PROOF. Substitute (13) and a similar formula for $k+1$ into the two sides respectively. After this substitution we observe that the number of terms is the same on both sides, since

$$\binom{n-m}{k-m}\binom{n}{k+1}\binom{k+1}{m} = \binom{n-m}{k+1-m}\binom{n}{k}\binom{k}{m}.$$

Also, the number of terms with a given $U = (\mu_1, \cdots, \mu_m)$ unaccented is the same, since

$$\binom{n-m}{k-m}\binom{n-m}{k+1-m} = \binom{n-m}{k+1-m}\binom{n-m}{k-m}.$$

Let the sum of all the terms with U unaccented in the two summations be denoted by $\sigma_{k+1} = \sigma_{k+1}(\mu_1, \cdots, \mu_m)$ and $\sigma_k = \sigma_k(\mu_1, \cdots, \mu_m)$ respectively. It is sufficient to prove that

$$\binom{n-m}{k-m} \sigma_{k+1} \leqq \binom{n-m}{k+1-m} \sigma_k, \tag{14}$$

for any U. σ_k contains $\binom{n-m}{k-m}$ terms each of the form $p_{\nu_1' \cdots \nu_l' \mu_1 \cdots \mu_m}$ where $0 \leqq l \leqq \mu_m - m$ and where $(\nu_1, \cdots, \nu_l, \mu_1, \cdots, \mu_m)$ is a $\binom{\mu_m}{m+l}$-combination out of $(1, \cdots, \mu_m)$. For fixed $(\mu_1, \cdots, \mu_m)$ and a fixed l but varying λ's, σ_k contains $\binom{n-\mu_m}{k-m-l}$ terms of the form $p_{\nu_1' \cdots \nu_l' \mu_1 \cdots \mu_m}$, with exactly l accented subscripts. Let the sum of all such terms be denoted by $\sigma_k^{(l)}$. Evidently $\sigma_k^{(l)}$ has $\binom{\mu_m - m}{l}$ terms. As a check we have

$$\binom{n-\mu_m}{k-m}\binom{\mu_m - m}{0} + \binom{n-\mu_m}{k-m-1}\binom{\mu_m - m}{1} + \cdots + \binom{n-\mu_m}{k-\mu_m}\binom{\mu_m - m}{\mu_m - m} = \binom{n-m}{k-m},$$

which is the total number of terms in σ_k.

We decompose these p's partially, as follows:

$$p_{\nu_1' \cdots \nu_l' \mu_1 \cdots \mu_m} = \sum_{b=0}^{\mu_m - m - l} \sum_{\mu_{m+1}, \cdots, \mu_{m+b}} p_{\nu_1' \cdots \nu_{l+c}' \mu_1 \cdots \mu_{m+b}},$$

where $(\nu_1, \cdots, \nu_{l+c}, \mu_1, \cdots, \mu_{m+b})$ is a permutation of $(1, \cdots, \mu_m)$ and where the second summation extends, for a fixed b, to all the $\binom{\mu_m - m - l}{b}$-combinations out of $(1, \cdots, \mu_m) - (\nu_1, \cdots, \nu_l, \mu_1, \cdots \mu_m)$.

Now consider a given

$$p_{\rho_1' \cdots \rho_t' \lambda_1 \cdots \lambda_s \mu_1 \cdots \mu_m}$$

where $0 \leqq t \leqq \mu_m - m$ and $(\rho_1 \cdots \rho_t \lambda_1 \cdots \lambda_s \mu_1 \cdots \mu_m)$ is a permutation of $(1, \cdots, \mu_m)$. It appears $\binom{t}{l}$ times in $\sigma_k^{(l)}$. Hence it appears

$$\binom{n-\mu_m}{k-m}\binom{t}{0} + \binom{n-\mu_m}{k-m-1}\binom{t}{1} + \cdots + \binom{n-\mu_m}{k-m-t}\binom{t}{t} = \binom{n-\mu_m+t}{k-m}$$

times in σ_k.

Therefore to prove (14) it is sufficient to prove that

$$\binom{n-m}{k-m}\binom{n-\mu_m+t}{k+1-m} \leqq \binom{n-m}{k+1-m}\binom{n-\mu_m+t}{k-m}.$$

By an easy reduction we have

$$(n - \mu_m + t - k + m) \leqq n - k$$

or

$$-\mu_m + t + m \leqq 0;$$

since $t \leqq \mu_m - m$ this is obvious.

THEOREM 4: *For $2 \leqq k \leqq n-1$ and $1 \leqq m \leqq k$ we have*

$$(5) \qquad \frac{\Sigma p_m(\nu_1, \cdots, \nu_k)}{\binom{n-m}{k-m}} \leqq \frac{1}{2}\frac{\Sigma p_m(\nu_1, \cdots, \nu_{k-1})}{\binom{n-m}{k-1-m}} + \frac{1}{2}\frac{\Sigma p_m(\nu_1, \cdots, \nu_{k+1})}{\binom{n-m}{k+1-m}}.$$

PROOF: By the reasoning in the previous proof, it is sufficient to show that

$$2\binom{n-m}{k-1-m}\binom{n-m}{k+1-m}\binom{n-\mu_m+t}{k-m} \leqq \binom{n-m}{k-m}\binom{n-m}{k+1-m}\binom{n-\mu_m+t}{k-1-m} + \binom{n-m}{k-m}\binom{n-m}{k-1-m}\binom{n-\mu_m+t}{k+1-m},$$

for $0 \leqq t \leqq \mu_m - m$. By an easy reduction this is equivalent to

$$2(n-k)(n-\mu_m+t-k+m+1) \leqq (n-k+1)(n-k) + (n-\mu_m+t-k+m+1)(n-\mu_m+t-k+m)$$

or

$$(n-\mu_m+t-k+m+1)(\mu_m-t-m) \leqq (n-k)(\mu_m-t-m).$$

For $t = \mu_m - m$ we have equality, otherwise we have

$$-\mu_m + t + m + 1 \leqq 0.$$

We can deduce Theorem 3 from Theorem 4 and the following result (a case of generalized Gumbel inequalities):

$$\binom{n-1}{n-2} p_m(1, \cdots, n) \leqq \Sigma p_m(\nu_1, \cdots, \nu_{n-1}). \tag{15}$$

PROOF OF (15): Substitute from (13). Consider the p's with U unaccented. The number of such terms is the same on both sides. But on the left-hand side they are all the same $p_{I'II'\cdots(U-1)'U}$, while those on the right-hand side, being of the form $p_{U_1'\cdots U_\lambda' U}$ where $0 \leqq \lambda \leqq U - 1$ and $(U_1, \cdots, U_\lambda)$ is a combination out of $(1, \cdots, U-1)$, are greater than or equal to it. Hence the result.

4. The $p_{\alpha_1\cdots\alpha_i}$'s in terms of the $p_m(\nu_1, \cdots, \nu_k)$'s and the $p_{[\alpha_1\cdots\alpha_i]}$'s in terms of the $p_1(\nu_1, \cdots, \nu_k)$'s.

THEOREM 5: *For* $1 \leqq m \leqq n$ *we have*

$$\begin{aligned} \binom{n-1}{m-1} p_{1\cdots n} &= \sum p_m(\nu_1, \cdots, \nu_m) - \sum p_m(\nu_1, \cdots, \nu_{m+1}) + \cdots \\ &\qquad + (-1)^{n-m} p_m(1, \cdots, n) \\ &= \sum_{i=0}^{n-m} (-1)^i \sum_{\nu_1, \cdots \nu_{m+i}} p_m(\nu_1, \cdots, \nu_{m+i}). \end{aligned} \tag{8}$$

PROOF: As in the proof of Theorem 3, consider $\sigma_k(\mu_1, \cdots, \mu_m)$. Here $m \leqq k \leqq n$. Since a given

$$p_{\rho_1'\cdots\rho_t'\lambda_1\cdots\lambda_s\mu_1\cdots\mu_m}, \tag{16}$$

appears $\binom{n-\mu_m+t}{k-m}$ times in σ_k, it appears

$$\begin{aligned} \sum_{k=m}^{n} (-1)^{k-m} \binom{n-\mu_m+t}{k-m} &= \sum_{j=0}^{n-m} (-1)^j \binom{n-\mu_m+t}{j} \\ &= \sum_{j=0}^{n-\mu_m+t} (-1)^j \binom{n-\mu_m+t}{j} = \begin{cases} 0, & \text{if } n-\mu_m+t \geqq 1, \\ 1, & \text{if } n-\mu_m+t = 0. \end{cases} \end{aligned}$$

times on the right hand side of (8). Hence for fixed $(\mu_1, \cdots, \mu_m)$, the only p's of the form (16) which actually appears are those with $t = \mu_m - n$. But $\mu_m \leqq n$, thus $t = 0$, $\mu_m = n$, and $(\lambda_1, \cdots \lambda_s, \mu_1, \cdots, \mu_m)$ is a permutation of $(1, \cdots, n)$. The term in question is therefore $p_{1\cdots n}$. Since the number of $\binom{n}{m}$-combinations of $(1, \cdots, n)$ with $\mu_m = n$ is $\binom{n-1}{m-1}$, we have the theorem.

THEOREM 6: *For* $1 \leqq r \leqq n-1$, *we have*

$$\begin{aligned} p_{[1\cdots r]} &= -p_1(r+1, \cdots, n) + \sum_{\nu_1} p_1(\nu_1, r+1, \cdots, n) \\ &\quad - \sum_{\nu_1,\nu_2} p(\nu_1, \nu_2, r+1, \cdots, n) + \cdots + (-1)^{r-1} \sum p_1(1, \cdots, n) \\ &= \sum_{i=1}^{r} (-1)^{i-1} \sum_{\nu_1, \cdots, \nu_i} p_1(\nu_1, \cdots, \nu_i, r+1, \cdots, n), \end{aligned} \tag{9}$$

where $(\nu_1, \cdots, \nu_i)$ *runs through all the* $\binom{r}{i}$*-combinations out of* $(1, \cdots, r)$.

PROOF: We rewrite (14) for the special case $m = 1$,

$$(17) \qquad p_1(\mu_1, \cdots, \mu_k) = p_{\mu_1} + p_{\mu_1'\mu_2} + \cdots + p_{\mu_1'\cdots\mu_{k-1}'\mu_k},$$

where $\mu_1 < \mu_2 < \cdots < \mu_k$. Substitute into the right hand side of (9). After the substitution let the sum of all those p's with μ unaccented be denoted by σ_μ. The terms in σ_μ are of the form $p_{\mu_1'\cdots\mu_{s-1}'\mu}$ where $1 \leqq s \leqq \mu$ and $(\mu_1, \cdots, \mu_{s-1})$ is a combination out of $(1, \cdots, \mu - 1)$.

First consider a fixed $\mu \leqq r$. For a fixed $p_{\mu_1'\cdots\mu_{s-1}'\mu}$ we count the number of times it appears in σ_μ, that is, on the right hand side of (9). This is evidently equal to

$$\sum_{j=s}^{r} (-1)^j \binom{r-\mu}{j-s} = \sum_{j=s}^{r-\mu+s} (-1)^j \binom{r-\mu}{j-s} = \begin{matrix} 0, & \text{if } r - \mu \geqq 1, \\ 1, & \text{if } r - \mu = 0. \end{matrix}$$

Thus the only terms that actually appear are those with $\mu = r$; and each of such terms $p_{\mu_1'\cdots\mu_{s-1}'r}$ appears exactly once with the sign $(-1)^s$. Hence their total contribution is

$$(18) \qquad p_r - \sum_{\nu_1} p_{\nu_1'r} + \sum_{\nu_1,\nu_2} p_{\nu_1'\nu_2'r} - \cdots + (-1)^{r-1} p_{1'\cdots(r-1)'r} = p_{1\cdots r},$$

by an easy modification of Poincaré's formula.

Next consider a fixed $\mu \geqq r + 1$. Every term with μ unaccented in σ_μ is of the form (with the usual convention for $\mu = r + 1$) $p_{\mu_1'\cdots\mu_s'(r+1)'\cdots(\mu-1)'\mu}$, where $(\mu_1, \cdots, \mu_s)$ is a combination out of $(1, \cdots, r)$; and it appears exactly once with the sign $(-1)^s$. Their total contribution is therefore

$$- p_{(r+1)'\cdots(\mu-1)'\mu} + \sum_{\nu_1} p_{\nu_1'(r+1)'\cdots(\mu-1)'\mu} - \sum_{\nu_1,\nu_2} p_{\nu_1'\nu_2'(r+1)'\cdots(\mu-1)'\mu} + \cdots$$
$$+ (-1)^{r-1} p_{1'\cdots(\mu+1)'\mu} = - p_{1\cdots r(r+1)'\cdots(\mu-1)'\mu},$$

by another application of Poincaré's formula. Summing up for $\mu = r + 1, \cdots, n$, we obtain

$$(19) \qquad -(p_{1\cdots r(r+1)} + p_{1\cdots r(r+1)'(r+2)} + \cdots + p_{1\cdots r(r+1)'\cdots(n-1)'n}).$$

Adding (18) and (19), we obtain as the sum of the right-hand side of (9)

$$p_{1\cdots r} - (p_{1\cdots r(r+1)} + p_{1\cdots r(r+1)'(r+2)} + \cdots + p_{1\cdots r(r+1)'\cdots(n-1)'n})$$
$$= p_{1\cdots r(r+1)'(r+2)'\cdots n'} = p_{[1\cdots r]}$$

by an easy modification of (17).

5. A condition for existence of systems of events associated with the probabilities $p_1(\nu_1, \cdots, \nu_k)$.

LEMMA 1: *Let any* $2^n - 1$ *quantities* $q(\alpha_1, \cdots, \alpha_k)$ *be given, where* $k =$

$1, \cdots, n$, *and for a fixed* k, $(\alpha_1, \cdots, \alpha_k)$ *runs through all the* $\binom{n}{k}$*-combinations out of* $(1, \cdots, n)$. *Let the quantities* $Q(\alpha_1, \cdots, \alpha_k)$ *be formed as follows:*

$$Q(0) = 1 - q(1, \cdots, n),$$

$$Q(\alpha_1, \cdots, \alpha_k) = -q(\alpha_{k+1}, \cdots, \alpha_n) + \sum_{\nu_1} q(\nu_1, \alpha_{k+1}, \cdots, \alpha_n) - \sum_{\nu_1, \nu_2} q(\nu_1, \nu_2, \alpha_{k+1}, \cdots, \alpha_n) + \cdots + (-1)^{k-1} q(1, \cdots, n),$$

where $(\nu_1, \cdots, \nu_i)$ runs through all the $\binom{k}{i}$-combinations out of $(1, \cdots, n) - (\alpha_{k+1}, \cdots, \alpha_n)$. *Then the sum of all these Q's is equal to* 1.

Proof: Add all these Q's and count the number of times a fixed $q(\mu_1, \cdots, \mu_k)$ appears in the sum. For $1 \leqq k \leqq n$ this number is equal to

$$-1 + \binom{k}{1} - \binom{k}{2} + \cdots + (-1)^{k-1}\binom{k}{k} = 0.$$

Hence we have the lemma.

Lemma 2: (Fréchet) *Given* 2^n *quantities* $Q_{[\alpha_1 \cdots \alpha_r]}$ *where* $(\alpha_1, \cdots, \alpha_r)$ *runs through all combinations out of* $(1, \cdots, n)$ *including the empty one. The necessary and sufficient condition that there exist systems of events* $E_1, \cdots, E_n$ *for which*

$$p_{[\alpha_1 \cdots \alpha_r]} = Q_{[\alpha_1 \cdots \alpha_r]}$$

(where $p_{[0]}$ denotes the probability for the non-occurrence of $E_1, \cdots, E_n$) *is that each* $Q \geqq 0$ *and that their sum is equal to* 1.

Proof: Since the probabilities $p_{[\alpha_1 \cdots \alpha_r]}$ are independent, i.e., unrelated in magnitudes except that their sum is equal to 1, the lemma is evident.

Theorem 7: *Given* $2^n - 1$ *quantities* $q(\alpha_1, \cdots, \alpha_k)$ *as in* Lemma 1, *the necessary and sufficient condition that there exist systems of events* $E_1, \cdots, E_n$ *for which*

$$p_1(\alpha_1, \cdots, \alpha_k) = q(\alpha_1, \cdots, \alpha_k)$$

is that for any combination $(\alpha_{r+1}, \cdots, \alpha_n)$, $1 \leqq r \leqq n-1$, *out of* $(1, \cdots, n)$ *we have*

$$-q(\alpha_{r+1}, \cdots, \alpha_n) + \sum_{\nu_1} q(\alpha_{\nu_1}, \alpha_{r+1}, \cdots, \alpha_n) - \sum_{\nu_1, \nu_2} q(\alpha_{\nu_1}, \alpha_{\nu_2}, \alpha_{r+1}, \cdots, \alpha_n) + \cdots + (-1)^{r-1} q(1, \cdots, n) \geqq 0,$$

and thus

$$1 - q(1, \cdots, n) \geqq 0.$$

Proof: The condition is necessary by Theorem 6. It is sufficient by Lemma 1, 2 and an obvious formula expressing $p_1(\alpha_1, \cdots, \alpha_r)$ in terms of the $p_{[\nu_1 \cdots \nu_i]}$'s.