ON FUNDAMENTAL SYSTEMS OF PROBABILITIES OF A FINITE NUMBER OF EVENTS

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We consider a probability function $P(E)$ defined over the Borel set of events generated by the n arbitrary events $E_1, \cdots, E_n$, which will be denoted by $\mathfrak{L}(1, \cdots, n)$.

We use the same notations as in the author's former paper[1], with the following abbreviations. We denote a combination $(\alpha_1 \cdots \alpha_a)$ simply by (α), and use the corresponding Latin letter a for its number of members. Similarly we write (β) for $(\beta_1 \cdots \beta_b)$, but (ν) for $(1, \cdots, n)$. We say that (β) belongs to (α) and write $(\beta) \epsilon (\alpha)$ when and only when the set $(\beta_1 \cdots \beta_b)$ is a subset of $(\alpha_1 \cdots \alpha_a)$. Then and then only we write $(\alpha) - (\beta)$ for the subset of elements of (α) that do not belong to (β); thus we may write it as (γ) with $c = a - b$. When and only when (α) and (β) have no common elements, we write $(\alpha) + (\beta)$ for the set of elements that belong either to (α) or to (β); thus we may write it as (γ), with $c = a + b \leqq n$. We note the case for empty sets: $(0) + (0) = (0)$. Now we can write $p_{[(\alpha)]}$ for $p_{[\alpha_1 \cdots \alpha_a]}$, $p_{((\alpha))}$ for $p_{\alpha_1 \cdots \alpha_a}$, $p_b((\alpha))$ for $p_b(\alpha_1 \cdots \alpha_a)$, etc. Further we denote by $p_{[b]}((\alpha))$ $(1 \leqq b \leqq a \leqq n)$ the probability of the occurrence of exactly b events out of $E_{\alpha_1}, \cdots, E_{\alpha_a}$, and write

$$P_a^{(m)}((\nu)) = \sum_{(\alpha) \epsilon (\nu)} p_m((\alpha)), \qquad P_a^{[m]}((\nu)) = \sum_{(\alpha) \epsilon (\nu)} p_{[m]}((\alpha));$$

since a is fixed by the left-hand sides, the summations on the right-hand sides are to be extended to all the $\binom{n}{a}$-combinations of (ν).

A sum written $\sum_{(\beta) \epsilon (\alpha)}$ is to be extended to all combinations (β), $b = 0, 1, \cdots, a$ belonging to (α), when b is not previously fixed; it is to be extended to all the $\binom{a}{b}$-combinations belonging to (α), when b is previously fixed.

Definition 1. *A system of quantities is said to form a fundamental system of probabilities for a set of events if and only if the probability of every event in the set can be expressed in terms of these quantities.*

Definition 2. *An event in $\mathfrak{L}(1, \cdots, n)$ is said to be symmetrical if and only if it is identical with every event obtained by interchanging any pair of suffixes (i, j) $(i, j = 1, \cdots, n)$ in the definition of it.* The subset of symmetrical events in $\mathfrak{L}(1, \cdots, n)$ will be denoted by $\mathfrak{S}(1, \cdots, n)$.

From the normal form[2] of every event in $\mathfrak{L}(1, \cdots, n)$ and the principle of

[1] "On the probability of the occurrence of at least m events among n arbitrary events," *Annals of Math. Stat.*, Vol. 12, 1941.

[2] See Hilbert-Ackermann, *Grundzüge der theoretischen Logik*, Chap. 1.

total probabilities, we can easily see the truth of the following theorems, which may of course be made more precise.

THEOREM. *The system of* $p_{[(\alpha)]}$, $(\alpha) \epsilon (\nu)$, 2^n *in number, forms a fundamental system for* $\mathfrak{L}(1, \cdots, n)$.

THEOREM. *The system of* $p_{[a]}((\nu))$, $0 \leqq a \leqq n$, $n + 1$ *in number, forms a fundamental system for* $\mathfrak{S}(1, \cdots, n)$.

Next, a theorem of Broderick[3], in a less precise form, may be stated:

The system of $p_{((\alpha))}$ $(p_{((0))} = 1)$, $(\alpha) \epsilon (\nu)$, 2^n *in number, forms a fundamental system for* $\mathfrak{L}$.

We may add in an easy way the following

THEOREM. *The system of* $S_a((\nu))$ $S_0((\nu)) = 1$, $0 \leqq a \leqq n$, $n + 1$ *in number, forms a fundamental system for* $\mathfrak{S}$.

In the present paper we shall prove, *inter alia*, the following four theorems of the above type, stated in more precise forms.

THEOREM 1. *For any E in* $\mathfrak{L}$, *we havc*

$$P(E) = c_0 + \sum_{\substack{(\alpha)\epsilon(\nu)\\ \alpha\neq 0}} c_\alpha p_1((\alpha)),$$

where $c_0 = 0$ *or* 1 *and the* c_α*'s are integers; and they are unique*[4].

THEOREM 2. *For any E in* $\mathfrak{S}$, *we have*

$$P(E) = c_0 + \sum_{a=1}^{n} c_a P_a^{(1)},$$

where $c_0 = 0$ *or* 1 *and the* c_a*'s are integers; and they are unique.*

THEOREM 3. *For any E in* $\mathfrak{L}$, *we have*

$$P(E) = d_0 + \sum_{\substack{(\alpha)\epsilon(\nu)\\ \alpha\neq 0}} d_\alpha p_{[1]}((\alpha)),$$

where $d_0 = 0$ *or* 1 *and the* d_α*'s are rational numbers and they are unique.*

THEOREM 4. *For any E in* $\mathfrak{S}$, *we have*

$$P(E) = d_0 + \sum_{a=1}^{n} d_a p_a^{[1]},$$

where $d_0 = 0$ *or* 1 *and the* d_a*'s are rational numbers; and they are unique.*

Less precisely, we may say that the system of $p_1((\alpha))$ or $p_{[1]}((\alpha))$ forms a fundamental system for $\mathfrak{L}$; the system of $P_a^{(1)}((\nu))$ or $P_a^{[1]}((\alpha))$ forms a fundamental system for $\mathfrak{S}$.

In fact however, we shall give much more than the mere proofs of

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[3] Fréchet, "Compléments à un théorème de T. S. Broderick concernant les événements dependants," *Proc. Edinburgh Math. Soc.*, Ser. 2, Vol. 6 (1939).

[4] "Unique" in the sense that it is impossible to replace therein the coefficients *c* by other numbers which are independent of the Borel set of events and the probability function.

these theorems. We shall establish the following explicit formulas for the general parameter m.

(1.1) (i) $p_{[(0)]} = 1 - p_1((\nu))$,

(ii) $p_{[(\alpha)]} = \sum_{\substack{(\beta)\epsilon(\alpha) \\ n-a+b\neq 0}} (-1)^{b-1} p_1((\nu) - (\alpha) + (\beta))$,[5] $\quad 1 \leq a \leq n.$

(1)
$$p_{[(\alpha)]} = (-1)^m \frac{m-1}{n-1} \sum_{c=m}^{n} \sum_{d=\max(0,c-a)}^{\min(c,n-a)} (-1)^{c-d} \binom{n-2}{a+d-m}^{-1} \sum_{\substack{(\delta)\epsilon(\nu)-(\alpha) \\ (\gamma)-(\delta)\epsilon(\alpha)}} p_m((\gamma) - (\delta) + (\delta)), \qquad n \geq a \geq m \geq 2.$$[5]

(2.1)
$$p_{[a]}((\nu)) = \sum_{\substack{b=n-a \\ b\neq 0}}^{n} (-1)^{b-n+a} \binom{b}{n-a} P_b^{(1)}((\nu)), \qquad 1 \leq a \leq n.$$

(2)
$$p_{[a]}((\nu)) = \sum_{b=m}^{n} (-1)^{b-m} L(n, a, b, m) P_b^{(m)}((\nu)), \qquad n \geq a \geq m \geq 2,$$

where

$$L(n, a, b, m) = \begin{cases} 0, & b < n - a + m - 1, \\ (-1)^{n-a} \binom{a}{m-1}^{-1} & b = n - a + m - 1, \\ \dfrac{(-1)^{n-a}(m-1)!(b-m)! \cdot (a-m)!\{ab - n(m-1\}}{a!(n-a)!(a+b-n-m+1)!}, & b > n - a + m - 1. \end{cases}$$

(3) (i)
$$p_{[(0)]}((\nu)) = 1 - \frac{1}{n} \sum_{c=1}^{n} \binom{n-1}{c-1}^{-1} P_c^{[1]}.$$

(ii)
$$p_{[(\alpha)]} = (-1)^m \frac{m}{n} \sum_{c=m}^{n} \sum_{d=\max(0,c-a)}^{\min(c,n-a)} (-1)^{c-d} \binom{n-1}{a+d-m}^{-1} \sum_{\substack{(\delta)\epsilon(\nu)-(\alpha) \\ (\gamma)-(\delta)\epsilon(\alpha)}} p_{[m]}((\gamma) - (\delta) + (\delta)), \qquad n \geq a \geq m \geq 1.$$

(4)
$$p_{[a]}((\nu)) = \sum_{b=m+n-a}^{n} (-1)^{n-a+b-m} \binom{b-m}{n-a} \binom{a}{m}^{-1} P_b^{[m]}((\nu)), \quad n \geq a \geq m \geq 1.$$

A simpler derivation of (1) than that given in an earlier paper[1] follows. Let us write Poincaré's formula as follows:

$$p_m((\beta)) = \sum_{c=m}^{b} (-1)^{c-m} \binom{c-1}{m-1} S_c((\beta)).$$

[5] Obviously we mean $((\nu) - (\alpha)) + (\beta)$ and $((\gamma) - (\delta)) + (\delta)$ respectively; similarly in the sequel.

Then for a fixed $b \geqq m$, summing over all $(\beta) \epsilon (\nu)$, we get

$$\sum_{(\beta)\epsilon(\nu)} p_m((\beta)) = \sum_{c=m}^{b} (-1)^{c-m} \binom{c-1}{m-1}\binom{n-c}{b-c} S_c((\nu)).$$

Hence

$$\begin{aligned}\sum_{b=m}^{n} (-1)^{b-m} \sum_{(\beta)\epsilon(\nu)} p_m((\beta)) &= \sum_{c=m}^{n} \binom{c-1}{m-1} S_c((\nu)) \sum_{b=c}^{n} (-1)^{b-c} \binom{n-c}{b-c} \\ &= \sum_{c=m}^{n} \binom{c-1}{m-1} S_c((\nu)) \begin{cases} 1 & \text{if } c = n \\ 0 & \text{if } c < n \end{cases} \\ &= \binom{n-1}{m-1} S_n((\nu)) = \binom{n-1}{m-1} p((\nu)).\end{aligned} \tag{1}$$

A change of notation gives, for $a + b \geqq m$,

$$\binom{a+b-1}{m-1} p_{((\alpha)+(\beta))} = \sum_{c=m}^{a+b} (-1)^{c-m} \sum_{(\gamma)\epsilon(\alpha)+(\beta)} p_m((\gamma)).$$

Hence

$$\begin{aligned}&\binom{a+b-1}{m-1} \sum_{(\beta)\epsilon(\nu)-(\alpha)} p_{((\alpha)+(\beta))} \\ &\quad = \sum_{c=m}^{a+b} (-1)^{c-m} \sum_{d=\max(0,c-a)}^{\min(c,n-a)} \binom{n-a-d}{b-d} \sum_{\substack{(\delta)\epsilon(\nu)-(\alpha) \\ (\gamma)-(\delta)\epsilon(\alpha)}} p_m((\gamma) - (\delta) + (\delta)).\end{aligned}$$

Substituting in the well-known formula, for $a \geqq 1$

$$p_{[(\alpha)]} = \sum_{b=0}^{n-a} (-1)^b \sum_{(\beta)\epsilon(\nu)-(\alpha)} p_{((\alpha)+(\beta))},$$

we get for $n \geqq a \geqq m$

$$\begin{aligned}p_{[(\alpha)]} &= \sum_{c=m}^{n} (-1)^{c-m} \sum_{d=\max(0,c-a)}^{\min(c,n-a)} \\ &\sum_{\substack{(\delta)\epsilon(\nu)-(\alpha) \\ (\gamma)-(\delta)\epsilon(\alpha)}} p_m((\gamma) - (\delta) + (\delta)) \left\{ \sum_{b=0}^{n-a} (-1)^b \binom{n-a-d}{b-d} \binom{a+b-1}{m-1}^{-1} \right\}.\end{aligned} \tag{1}$$

Thus the problem reduces to the summation of the following series:

$$\sum_{b=0}^{n-a} (-1)^b \binom{n-a-d}{b-d} \binom{a+b-1}{m-1}^{-1}.$$

Case 1: $m = 1$. In this case the series reduces to

$$\sum_{b=0}^{n-a} (-1)^b \binom{n-a-d}{b-d} = \begin{cases} (-1)^{n-a} & \text{if } d = n - a, \\ 0 & \text{if } d < n - a. \end{cases}$$

Hence for $a \geqq 1$,

$$p_{[(\alpha)]} = \sum_{c=\max(1,n-a)}^{n} (-1)^{c-1} \sum_{(\gamma)-((\nu)-(\alpha))\epsilon(\alpha)} p_1((\nu)-(\alpha)+(\gamma)-((\nu)-(\alpha)))(-1)^{n-a}$$

Writing $(\gamma) - ((\nu) - (\alpha)) = (\beta)$, we obtain

$$p_{[(\alpha)]} = \sum_{b=\max(1-n+a,0)}^{a} (-1)^{b-1} \sum_{(\beta)\,\epsilon\,(\alpha)} p_1((\nu) - (\alpha) + (\beta)).$$

This is equivalent to (1.1), (ii), while (i) is trivial.

Case 2: $m \geqq 2$. We have, for $c \geqq 1$,

$$\sum_{l=0}^{a} (-1)^l \binom{a}{l}\binom{b+l}{c}^{-1} = \frac{c}{a+b}\binom{a+b-1}{b-c}^{-1},$$

which is easily proved by induction on a.

Hence for $m \geqq 2$,

$$\sum_{b=0}^{n-a} (-1)^b \binom{n-a-d}{b-d}\binom{a+b-1}{m-1}^{-1}$$
$$= \sum_{b'=-d}^{n-a-d} (-1)^{d+b'}\binom{n+a-d}{b'}\binom{a+b'+d-1}{m-1}^{-1}$$
$$= (-1)^d \sum_{b'=0}^{n-a-d} (-1)^{b'}\binom{n-a-d}{b'}\binom{a+d-1+b'}{m-1}^{-1}$$
$$= (-1)^d \frac{m-1}{n-1}\binom{n+2}{a+d-m}^{-1}$$

Substituting in (1) we get formula (1).

To derive formula (2.1) for a fixed a, $1 \leqq a \leqq n$, we sum (1.1, ii), which gives

$$p_{[a]}((\nu)) = \sum_{(\alpha)\,\epsilon\,(\nu)} p_{[(\alpha)]} = \sum_{\substack{b=0\\ n-a+b\neq 0}}^{a} (-1)^{b-1} \sum_{(\alpha)\,\epsilon\,(\nu)} \sum_{(\beta)\,\epsilon\,(\alpha)} p_1((\nu) - (\alpha) + (\beta)).$$

Letting $(\nu) - (\alpha) + (\beta) = (\gamma)$, we get

$$p_{[a]}((\nu)) = \sum_{c=\max(1,n-a)}^{n} (-1)^{n-a+c-1}\binom{c}{n-a}\sum_{(\gamma)\,\epsilon\,(\nu)} p_1((\gamma)),$$

which is formula (2.1).

The following form of Poincaré's formula is of assistance in deriving (2):

$$p_{[a]}((\nu)) = \sum_{c=a}^{n} (-1)^{c-a}\binom{c}{a} S_a((\nu)).$$

Substituting from (1), we get

$$p_{[a]}((\nu)) = \sum_{c=a}^{n} (-1)^{c-a} \binom{c}{a} \binom{c-1}{m-1}^{-1} \sum_{b=m}^{c} (-1)^{b-m} \binom{n-b}{c-b} P_b^{(m)}((\nu))$$

$$= \sum_{b=m}^{n} (-1)^{b-m} P_b^{(m)}((\nu)) \left\{ \sum_{c=\max(a,b)}^{n} (-1)^{c-a} \binom{c}{a} \binom{n-b}{c-b} \binom{c-1}{m-1}^{-1} \right\}.$$

Thus the problem reduces to the summation of the following series:

$$L(n, a, b, m) = \sum_{c=\max(a,b)}^{n} (-1)^{c-a} \binom{c}{a} \binom{n-b}{c-b} \binom{c-1}{m-1}^{-1}$$

First, we have, for $z \geqq 0$, $y \geqq w$,

$$\sum_{x=\max(0,1-w)}^{z} (-1)^x \binom{z}{x} (x+y) \cdots (x+w)$$

$$= \begin{cases} 0 & \text{if } y - w + 1 < z, \\ \dfrac{(-1)^z y!(y+1-w)!}{(z+w-1)!(y+1-w-z)!} & \text{if } y - w + 1 \geqq z, \end{cases}$$

which may be easily proved by induction on z.

Next, we have

$$L(n, a, b, m) = \frac{(m-1)!}{a!} \sum_{c=\max(a,b)}^{n} (-1)^{c-a} \binom{n-b}{c-b} \frac{c(c-m)!}{(c-a)!}$$

$$= \frac{(m-1)!}{a!} \sum_{c'=\max(0,a-b)}^{n-b} (-1)^{c'+b-a} \binom{n-b}{c'} \frac{(c'+b)(c'+b-m)!}{(c'+b-a)!}$$

$$= (-1)^{b-a} \frac{(m-1)!}{a!} \sum_{c'=\max(0,a-b)}^{n-b} (-1)^{c'}$$

$$\cdot \binom{n-b}{c'} \frac{(c'+b-m+1)! + (m-1)(c'+b-m)!}{(c'+b-a)!}$$

$$= (-1)^{b-a} \frac{(m-1)!}{a!} \{T(n, a, b, m) + (m-1)T(n, a, b, m+1)\},$$

where

$$T(n, a, b, m) = \sum_{c=\max(0,a-b)}^{n-b} (-1)^c \binom{n-b}{c} \frac{(c+b-m+1)!}{(c+b-a)!}$$

$$= \begin{cases} 0 & \text{if } b < n - a + m - 1, \\ \dfrac{(-1)^{n-b}(a-m+1)!(b-m+1)!}{(n-a)!(a+b-n-m+1)!} & \text{if } b \geqq n - a + m - 1, \end{cases}$$

by the preceding formula. Thus we get the explicit expression for $L(n, a, b, m)$ given in formula (2), which is thereby proved.

The derivations of formulas 3 and 4 are similar to the above and may be omitted.

Now we can give the essential argument for Theorems 1–4. It is evident that for any E in $\mathfrak{L}$, we have

$$P(E) = \Sigma p_{[(\alpha)]},$$

where the summation extends to certain combinations $(\alpha) \epsilon (\nu)$. Substituting from formula (1.1) we get Theorem 1; substituting from formula (3) we get Theorem 3. Next, for any E in $\mathfrak{S}$, we have

$$P(E) = \Sigma p_{[a]}((\nu)),$$

where the summation extends to certain values of a. Substituting from formula (1.1), (i) and formula (2) we get Theorem 2; substituting from formula (3), (i) and formula (4) we get Theorem 4. We may note these proofs are "constructive".

It remains to prove the uniqueness of the coefficients in Theorems 1–4. For Broderick's theorem this has been done by Fréchet[3], by introducing "independent events". Our proof will be based on the conditions of existence, also initiated by Fréchet[6], for the systems $p_1((\alpha))$, $p_{[1]}((\alpha))$, $P_a^{(1)}((\nu))$, $P_a^{[1]}((\nu))$.

The conditions of existence of the system $p_1((\alpha))$ have been given by the author in the paper[1], though the proof there is not quite complete.

1. *Conditions of existence of the system* $P_a^{(1)}((\nu))$. Given n quantities $Q_a^{(1)}$, $1 \leqq a \leqq n$; what are the necessary and sufficient conditions that they may be the system of $P_a^{(1)}((\nu))$'s, $1 \leqq a \leqq n$, of a probability function defined over $\mathfrak{S}(1, \cdots, n)$?

From formula (1.1), (i) and formula (2) it is evident that necessary conditions are, for $1 \leqq a \leqq n$,

$$\sum_{\substack{b=n-a \\ b\neq 0}}^{n} (-1)^{b-n+a-1} \binom{b}{n-a} Q_b^{(1)} \geqq 0, \tag{3}$$

$$1 - Q_n^{(1)} \geqq 0,$$

and

$$\sum_{a=1}^{n} \sum_{\substack{b=n-a \\ b\neq 0}}^{n} (-1)^{b-n+a-1} \binom{b}{n-a} Q_b^{(1)} + 1 - Q_n^{(1)} = 1. \tag{4}$$

The last condition can be re-written as

$$\sum_{b=1}^{n} (-1)^{b-1} Q_b^{(1)} \sum_{a=\max(1,n-b)}^{n} (-1)^{n-a} \binom{b}{n-a} + 1 - Q_n^{(1)} = 1,$$

which reduces to the identity $1 = 1$.

[6] "Conditions d'existence de système d'événements associés à certaines probabilités," *Jour. de Math.*, 1940. However, our interpretation of the term would mean instead "conditions of existence of a probability function defined over a Borel set of events, etc."

To show that the conditions (3) are sufficient, put

$$p_{[a]} = \sum_{b=n-a}^{n} (-1)^{b-n+a-1}\binom{b}{n-a} Q_b^{(1)},$$

$$p_{[0]} = 1 - Q_n^{(1)}.$$

By (3) and (4) we have, for $0 \leqq a \leqq n$,

$$p_{[a]} \geqq 0 \quad \text{and} \quad \sum_{a=0}^{n} p_{[a]} = 1.$$

Hence they are actually the $p_{[a]}((\nu))$'s of a probability function. We want to show that the $P_a^{(1)}((\nu))$'s of this probability function coincide with the given $Q_a^{(1)}$'s, so that this is the probability function we seek. We have,

$$P_b^{(1)}((\nu)) = \sum_{(\beta)\,\epsilon\,(\nu)} p_1((\beta)) = \sum_{a=1}^{n} p_{[a]} \sum_{h=\max(1,b-n+a)}^{\min(a,b)} \binom{a}{h}\binom{n-a}{b-h}$$

$$= \sum_{c=0}^{n} \left\{ \sum_{a=\max(1,n-c)}^{n} (-1)^{c-n+a-1}\binom{c}{n-a} \sum_{h=\max(1,b-n+a)}^{\min(a,b)} \binom{a}{h}\binom{n-a}{b-h} \right\} Q_c^{(1)}.$$

Now the series in curl brackets

$$= \sum_{a=\max(1,n-c)}^{n-b} (-1)^{c-n+a-1}\binom{c}{a}\left\{\binom{n}{b} - \binom{n-a}{b}\right\}$$

$$+ \sum_{a=n-b+1}^{n} (-1)^{c-n+a-1}\binom{c}{n-a}\binom{n}{b}$$

$$= \sum_{a=\max(1,n-c)}^{n} (-1)^{c-n+a-1}\binom{c}{n-a}\binom{n}{b}$$

$$- \sum_{a=\max(1,n-c)}^{n-b} (-1)^{c-n+a-1}\binom{c}{n-a}\binom{n-a}{b}.$$

If $c = n$, the last

$$= \binom{n}{b} - \sum_{a=1}^{n-b} (-1)^{a-1}\binom{n}{n-b}\binom{n-b}{a}$$

$$= \binom{n}{b} - \binom{n}{b}\sum_{a=1}^{n-b} (-1)^{a-1}\binom{n-b}{a} = \begin{cases} 1 & \text{if} \quad b = n; \\ 0 & \text{if} \quad b \neq n. \end{cases}$$

If $c < n$, we have

$$= 0 + (-1)^c \sum_{a=n-c}^{n-b} (-1)^{n-a}\binom{c}{n-a}\binom{n-a}{b}$$

$$= (-1)^c \sum_{a'=c}^{b} (-1)^{a'}\binom{c}{a'}\binom{a'}{b} = \begin{cases} 1 & \text{if} \quad b = c; \\ 0 & \text{if} \quad b \neq c. \end{cases}$$

Therefore

$$P_b^{(1)}((\nu)) = Q_b^{(1)}.$$

2. *Conditions of existence of the system* $p_{[1]}((\alpha))$. Given $2^n - 1$ quantities $q_{[1]}((\alpha))$, $(\alpha) \epsilon (\nu)$, $a \geqq 1$, what are the necessary and sufficient conditions that they may be the system of $p_{[1]}((\alpha))$'s, of a probability function defined over $\mathfrak{L}(1, \cdots, n)$?

From formula 3 it is evident that necessary conditions are

$$\frac{1}{n}\sum_{c=1}^{n}\sum_{d=\max(0,c-a)}^{\min(c,n-a)}(-1)^{c-d-1}\binom{n-1}{a+d-1}^{-1}\sum_{\substack{(\delta)\,\epsilon\,(\nu)-(\alpha)\\(\gamma)-(\delta)\,\epsilon\,(\alpha)}} q_{[1]}((\gamma)-(\delta)+(\delta)) \geqq 0, \tag{5}$$

$$1-\frac{1}{n}\sum_{c=1}^{n}\binom{n-1}{c-1}^{-1}\sum_{(\gamma)\,\epsilon\,(\nu)} p_{[1]}((\gamma)) \geqq 0;$$

and

$$1+\frac{1}{n}\sum_{(\alpha)\epsilon(\nu)}\sum_{c=1}^{n}\sum_{d=\max(0,c-a)}^{\min(c,n-a)}(-1)^{c-d-1}\binom{n-1}{a+d-1}^{-1}\sum_{\substack{(\delta)\epsilon(\nu)-(\alpha)\\(\gamma)-(\delta)\epsilon(\alpha)}} q_{[1]}((\gamma)-(\delta)+(\delta)) = 1. \tag{6}$$

Consider the sum

$$\sum_{(\alpha)\epsilon(\nu)}\sum_{d=\max(0,c-a)}^{\min(c,n-a)}(-1)^{d}\binom{n-1}{a+d-1}^{-1}\sum_{\substack{(\delta)\epsilon(\nu)-(\alpha)\\(\gamma)-(\delta)\epsilon(\alpha)}} q_{[1]}((\gamma)-(\delta)+(\delta)).$$

For a fixed (δ), the number of ways of writing $(\gamma) = (\gamma) - (\delta) + (\delta)$ is $\binom{c}{d}$, then since $(\gamma) - (\delta) \epsilon (\alpha)$ but $(\alpha) - ((\gamma) - (\delta)) \epsilon (\nu) - (\gamma)$, the number of choices of (α) is $\binom{n-c}{a-c+d}$. Thus the coefficient of $q_{[1]}((\gamma))$ in the sum is

$$\sum_{a=0}^{n}\sum_{d=\max(0,c-a)}^{\min(c,n-a)}(-1)^{d}\binom{c}{d}\binom{n-c}{a-c+d}\binom{n-1}{a+d-1}^{-1}$$
$$=\binom{n-1}{c-1}^{-1}\sum_{a=0}^{n}\sum_{d=\max(0,c-a)}^{\min(c,n-a)}(-1)^{d}\binom{c}{d}\binom{a+d-1}{c-1}=0.$$

Therefore the condition (6) reduces to the identity $1 = 1$.

To show that conditions (6) are sufficient, put the left-hand sides of (5) equal to $p_{[(\alpha)]}$ and $p_{[(0)]}$ respectively. Then

$$p_{((\alpha))} = \sum_{(\beta)\epsilon(\nu)-(\alpha)} p_{[(\alpha)+(\beta)]}$$
$$= \frac{1}{n}\sum_{c=1}^{n}(-1)^{c-1}\sum_{b=0}^{n-a}\sum_{d=\max(0,c-a-b)}^{\min(c,n-a-b)}(-1)^{d}\binom{n-1}{a+d-1}^{-1}\sum_{(\beta)\epsilon(\nu)-(\alpha)}\sum_{\substack{(\delta)\epsilon(\nu)-(\alpha)-(\beta)\\(\gamma)-(\delta)\epsilon(\alpha)+(\beta)}} q_{[1]}((\gamma)-(\delta)+(\delta)). \tag{7}$$

Let $(\gamma) = (\gamma) - (\phi) + (\phi)$, where $(\phi) \in (\alpha)$, $(\gamma) - (\phi) \in (\nu) - (\alpha)$. Then the sum in the curl brackets can be written, by a combinatorial calculation, as

$$\sum_{f=0}^{\min(a,c)} \left\{ \sum_{b=0}^{n-a} \sum_{d=\max(0,c-f-b)}^{\min(c-f,n-a-b)} (-1)^d \binom{c-f}{d} \binom{n-a-c+f}{b-c+d+f} \binom{n-1}{a+b+d-1}^{-1} \right\} \sum_{\substack{(\phi)\in(\alpha) \\ (\gamma)-(\phi)\in(\nu)-(\alpha)}} q_{[1]}((\gamma)-(\phi)+(\phi)).$$

The sum in the last curl brackets is

$$\binom{n-1}{a+c-f-1}^{-1} \sum_{b=0}^{n-a} \sum_{d=\max(0,c-f-b)}^{\min(c-f,n-a-b)} (-1)^d \binom{c-f}{d} \binom{a+b+d-1}{a+c-f-1}.$$

Inverting the order of summations,

$$\binom{n-1}{a+c-f-1}^{-1} \sum_{d=\max(0,c-f-n+a)}^{\min(c-f,n-a)} (-1)^d \binom{c-f}{d} \sum_{b=c-f-d}^{n-a-d} \binom{a+b+d-1}{a+c-f-1}$$

$$= \binom{n-1}{a+c-f-1}^{-1} \sum_{d=\max(0,c-f-n+a)}^{\min(c-f,n-a)} (-1)^d \binom{c-f}{d} \binom{n}{a+c-f}$$

$$= \binom{n}{a+c-f} \binom{n-1}{a+c-f-1}^{-1} \sum_{d=0}^{c-f} (-1)^d \binom{c-f}{d} = \begin{cases} \dfrac{n}{a} & \text{if } f = c, \\ 0 & \text{if } f \neq c. \end{cases}$$

Hence (7) reduces to

$$p_{((\alpha))} = \frac{1}{a} \sum_{c=1}^{n} (-1)^{c-1} \sum_{(\gamma)\in(\alpha)} q_{[1]}((\gamma)).$$

Then

$$S_b((\alpha)) = \sum_{(\beta)\in(\alpha)} p_{((\beta))} = \frac{1}{b} \sum_{\substack{(\delta)\in(\alpha) \\ d\neq 0}} (-1)^{d-1} \binom{a-d}{b-d} q_{[1]}((\delta))$$

$$p_{[1]}((\alpha)) = \sum_{b=1}^{a} (-1)^{b-1} S_b((\alpha))$$

$$= \sum_{\substack{(\delta)\in(\alpha) \\ d\neq 0}} \left\{ \sum_{b=1}^{a} (-1)^{b-d} \binom{a-d}{b-d} \right\} q_{[1]}((\delta)) = q_{[1]}((\alpha)).$$

The conditions of existence of the system $P_a^{[1]}((\nu))$, $1 \leqq a \leqq n$, are similarly deduced from formula (3), (i) and formula (4) with $m = 1$.

Now we can prove the uniqueness of the coefficients in Theorems 1–4. Since the proofs are all exactly similar, we take Theorem 2. Suppose, if possible, there exists another system of coefficients c'_a, $0 \leqq a \leqq n$ so that

$$P(E) = c_0 + \sum_{a=1}^{n} c_a P_a^{(1)}((\nu)) = c'_0 + \sum_{a=1}^{n} c'_a P_a^{(1)}((\nu)).$$

Taking the difference, we get a linear polynomial in the variables $P_a^{(1)}((\nu))$, $1 \leqq a \leqq n$ which must vanish:

$$(8) \qquad (c_0 - c_0') + \sum_{a=1}^{n} (c_a - c_a')P_a^{(1)}((\nu)) = 0,$$

for all "admissible" values of the variables. These values, say $Q_a^{(1)}$, are precisely those which satisfy the conditions (3).

It is evidently easy to construct a system of $Q_a^{(1)}$, $1 \leqq a \leqq n$, which satisfy the conditions (3) written with the sign of strict inequality "$>$". Hence in a sufficiently small neighborhood of the point $(Q_1^{(1)}, Q_2^{(1)}, \cdots, Q_n^{(1)})$ in the n-dimensional space these strict inequalities still hold. Hence the polynomial vanishes in this neighborhood and so must vanish identically; that is,

$$c_a - c_a' = 0 \quad \text{for} \quad 0 \leqq a \leqq n. \qquad \text{Q. E. D.}$$