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THE CONDENSER PROBLEM

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The condenser theorem in classical potential theory is studied within the framework of Markov processes and probabilistic potential theory. The condenser charge is expressed in terms of successive balayages of a capacitary measure.

1. Introduction. In classical potential theory on $\mathbb{R}^d$ with $d \geqq 3$ (or, more generally, in theory of Dirichlet spaces) the "condenser theorem" states the the following (see, for example, page 380 of [5]). Let G_0 and G_1 be open sets with disjoint closures $\bar{G}_0$ and $\bar{G}_1$ and assume that $\bar{G}_1$ is compact. Then there exists a potential p of a signed measure ν such that:

(i) $0 \leqq p \leqq 1$ a.e. on $\mathbb{R}^d$.

(ii) $p = 0$ a.e. on G_0 and $p = 1$ a.e. on G_1.

(iii) The support of ν^+ is contained in $\bar{G}_1$ and the support of ν^- is contained in $\bar{G}_0$.

From (i) and (ii) one would guess that $p(x)$ is just the probability that a Brownian motion starting at x hits G_1 before G_0, and consequently (i) and (ii) hold everywhere rather than almost everywhere. With this motivation it is very easy to give a probabilistic proof of the condenser theorem and to study the condenser problem within the framework of Markov processes. This note is devoted to such a study. In order to keep things simple we shall consider only Hunt processes with a locally compact metrizable state space E. (The expert should have no difficulty extending our results to the "right" processes.) Our method yields some interesting by-products. For example, it turns out that ν^+ is the capacitary measure, μ, of G_1 for the process killed when it first hits G_0 and that ν^- is the balayage of $\nu^+ = \mu$ on G_0. Moreover, we obtain an explicit formula (3.2) for μ in terms of the successive balayages on G_0 and G_1 of the capacitary measure π of G_1 for the entire process.

2. Let X be a Hunt process with state space E as in [2]. We refer the reader to [2] for all unexplained notation and terminology. Let D and B be nearly Borel sets with disjoint closures. We assume that D is transient in the sense that if $L = L_D = \sup\{t: X_t \in D\}$, then $L < \infty$ almost surely. (By convention the supremum of the empty set is zero and the infimum of the empty set is infinity.) As

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usual $T_D = \inf\{t > 0 : X_t \in D\}$ denotes the hitting time of D. Let

$$\varphi(x) = P_D 1(x) = P^x(T_D < \infty) = P^x(L > 0)\,, \tag{2.1}$$
$$p(x) = P^x(T_D < T_B)\,.$$

Then φ is an excessive function, while p is excessive relative to (X, T_B). See Section III-5 of [2]. The operators P_D and P_B are the usual balayage or hitting operators. An inclusion-exclusion argument leads to the following formula

$$p = P^{\bullet}(T_D < T_B) = P_D 1 - P_B P_D 1 + P_D P_B P_D 1 - \cdots .$$

The next proposition makes this precise. (C. Nevison informed us that he used it in a prior discussion.)

(2.2) PROPOSITION. *Let* $p_n = (P_D P_B)^n P_D 1 = (P_D P_B)^n \varphi$. *Then*

$$p = \textstyle\sum_{n=0}^{\infty} (p_n - P_B p_n)\,.$$

PROOF. Each p_n is excessive, bounded by one, and $P_B p_n \leqq p_n$. Therefore $0 \leqq p_n - P_B p_n \leqq 1$. Let $T_0 = 0$, $T_1 = T_D$, $T_2 = T_D + T_B \circ \theta_{T_D}$, $\cdots$, $T_{2n+1} = T_{2n} + T_D \circ \theta_{T_{2n}}$, $T_{2n+2} = T_{2n+1} + T_B \circ \theta_{T_{2n+1}}$. Thus $T_1, T_2, T_3, \cdots$ are the times of the successive visits to D, then to B, then back to D, and so on. A simple induction shows that $P_{T_{2n}} = (P_D P_B)^n$ for each $n \geqq 0$. It is straightforward to check that

$$P^x\{T_{2n+1} \leqq L \leqq T_{2n+2};\, T_D < T_B\} = p_n(x) - P_B p_n(x)$$

because L must lie in one of the intervals $[T_{2n+1}, T_{2n+2}]$. Note that the quasi-left-continuity of X implies that $\lim_n T_n = \infty$. This completes the proof of (2.2).

If $\sum p_n$ converges, then (2.2) may be written in the more agreeable form

$$p = \sum p_n - \sum P_B p_n\,. \tag{2.3}$$

We shall give some simple conditions that guarantee the convergence of $\sum p_n$. The hypotheses on D and B in the first paragraph of this section are still in force.

(2.4) PROPOSITION. *Suppose there exists a nearly Borel set G with $D \subset G \subset B^c$ and satisfying*:

(i) $\sup\{U(x, G) : x \in E\} = M < \infty$.

(ii) *There exist $t_0 > 0$ and $\eta > 0$ such that $P^x(T_{G^c} \geqq t_0) \geqq \eta$ for all $x \in \bar{D}^f$—the fine closure of D.*

Then $\sum p_n(x)$ is bounded in x.

PROOF. Let (T_n) be the sequence defined in the proof of Proposition 2.2. Then

$$p_n(x) = P_{T_{2n}} \varphi(x) = P_{T_{2n}} P_D 1(x) = P^x(T_{2n+1} < \infty)$$

for each $n \geqq 0$. Since $L < \infty$ and $T_n \uparrow \infty$ it is obvious that

$$P^x(T_{2n+1} < \infty \text{ for all } n) = 0\,.$$

Thus (2.4) is a matter of strenghtening this trivial fact to

$$\sup \textstyle\sum_n P^x(T_{2n+1} < \infty) < \infty\,.$$

If $y \in \bar{D}^f$, then by (ii)

$$E^y \int_0^{T_{G^c}} 1_G(X_t)\,dt \geqq \eta t_0 .$$

Now using (i) we have

$$M \geqq U(x, G) \geqq \sum_{n=0}^{\infty} E^x \int_{T_{2n+1}}^{T_{2n+2}} 1_G(X_t)\,dt$$
$$= \sum_{n=0}^{\infty} E^x\{E^{X(T_{2n+1})} \int_0^{T_B} 1_G(X_t)\,dt\} .$$

But $T_B \geqq T_{G^c}$ and $X(T_{2n+1}) \in \bar{D}^f$ if $T_{2n+1} < \infty$. Therefore

$$M \geqq \eta t_0 \sum_{n=0}^{\infty} P^x(T_{2n+1} < \infty) ,$$

establishing (2.4).

REMARKS. In (2.4 ii) one need only assume that $g(x) = P^x(T_{G^c} \geqq t_0) \geqq \eta$ for $x \in D$ because it is immediate from (II-4.14) of [2] that g is finely continuous. If in (2.4 i) one only assumes that $U(x, G)$ is finite for each x, then the proof shows that $\sum p_n(x)$ is finite for each x.

We next formulate a simple condition under which the hypotheses of (2.4) hold. The basic result that we need is a "separation" lemma that holds when the semigroup (P_t) maps $\mathbf{C}_0$ into $\mathbf{C}_0$. Here $\mathbf{C}_0$ is the space of continuous functions on E that vanish at infinity. This result is well known and may be found in [1], for example. Nevertheless we shall give the simple proof for the convenience of the reader.

(2.5) LEMMA. *Let (P_t) map $\mathbf{C}_0$ into $\mathbf{C}_0$. Let K be compact and let G be an open neighborhood of K. Then for each $\delta > 0$ there exists a $t_0 > 0$ such that*

$$\inf_{x \in K} P^x(T_{G^c} \geqq t_0) \geqq 1 - \delta . \tag{2.6}$$

PROOF. We may assume without loss of generality that G has compact closure. For typographical convenience let $T = T_{G^c}$ during this proof. Since (P_t) maps $\mathbf{C}_0$ into $\mathbf{C}_0$ and $P_t f \to f$ pointwise as $t \to 0$ for each $f \in \mathbf{C}_0$, it follows that, in fact, $||P_t f - f|| \to 0$ as $t \to 0$ for each $f \in \mathbf{C}_0$ where $||\cdot||$ is the usual supremum norm. See, for example, II-(2.15) of [2]. Choose $f \in \mathbf{C}_0$ with $0 \leqq f \leqq 1$, $f = 1$ on K, and $f = 0$ on G^c. Given $\delta > 0$ there exists $t_0 > 0$ such that $||P_t f - f|| < \delta/2$ for all $t \leqq t_0$. Therefore

$$\sup_{t \leq t_0} \sup_{x \notin G} P_t f(x) < \delta/2 \tag{2.7}$$

$$\inf_{t \leq t_0} \inf_{x \in K} P_t f(x) > 1 - \delta/2 . \tag{2.8}$$

Thus if $x \in K$

$$1 - \delta/2 < E^x[f \circ X_{t_0}] \leqq P^x[T \geqq t_0] + E^x[f \circ X_{t_0}; T < t_0] , \tag{2.9}$$

and the strong Markov property implies

$$E^x[f \circ X_{t_0}; T < t_0] = E^x\{E^{X(T)}[f \circ X_{(t_0 - T)^+}]; T < t_0\} .$$

But $X(T) \in G^c$ if $T < \infty$ and so by (2.7) this last expectation does not exceed $\delta/2$. Combining this with (2.9) yields

$$1 - \delta/2 \leqq \inf_{x \in K} P^x(T \geqq t_0) + \delta/2 ,$$

completing the proof of (2.5).

The following corollary is an immediate consequence of (2.4) and (2.5). Here, of course, B and D satisfy the conditions in the first paragraph of this section.

(2.10) COROLLARY. *Let* (P_t) *map* $\mathbf{C}_0$ *into* $\mathbf{C}_0$ *and assume that* X *is transient in the sense that* $x \to u\,(x, K)$ *is bounded for each compact* K. *Then if* D *has compact closure* $\sum p_n(x)$ *is bounded in* x *and*

$$p = \sum p_n - \sum P_B p_n .$$

3. In this section we shall assume that X satisfies the duality assumptions in Section VI-1 of [2] and the mild transience condition that there exists a sequence (h_n) of nonnegative functions with $h_n \uparrow 1$ and Uh_n finite for each n. Then for each x the potential kernel $u(x, y)$ is finite almost everywhere in y. See Section VI-1 of [2] for notation and terminology. As in the previous sections B and D are nearly Borel sets with disjoint closures with $L_D < \infty$. In addition throughout this section we shall suppose that the capacitary measure π_D of D exists; that is, π_D is the unique measure carried by $\bar{D}$ satisfying $\varphi = P_D 1 = U\pi_D$. For example, if $\bar{D}$ is compact and X satisfies conditions (VI-2.1), (VI-2.2), (VI-4.1), and (VI-4.2) of [2], then π_D exists. (See (VI-4.3) of [2].) However, much weaker conditions suffice. See [3] or [6] in this connection.

Let $v(x, y)$ be the potential kernel for (X, T_B)—the process X killed when it first hits B. Then v is positive kernel satisfying

$$u(x, y) = v(x, y) + P_B u(x, y) = v(x, y) + u\hat{P}_B(x, y) . \tag{3.1}$$

See [4], for example. As usual, write $V\mu(x) = \int v(x, y)\mu(dy)$ when μ is a positive measure. Let $\mu_n = \sum_{k \le n} (\hat{P}_D \hat{P}_B)^k \pi_D$, and

$$\mu_D = \lim_n \mu_n = \sum_{k=0}^{\infty} (\hat{P}_D \hat{P}_B)^k \pi_D . \tag{3.2}$$

Then μ_D is a positive measure carried by $\bar{D}$ since each μ_n is carried by $\bar{D}$. Of course, a priori, μ_D need not have any reasonable finiteness properties. However, V is a positive kernel and so

$$V\mu_D(x) = \lim V\mu_n(x)$$

exists. The fundamental identity for dual processes, VI-(1.16) of [2], yields

$$U\mu_n = \sum_{k=0}^{n} U(\hat{P}_D \hat{P}_B)^k \pi_D = \sum_{k=0}^{n} (P_D P_B)^k U\pi_D = \sum_{k=0}^{n} p_k . \tag{3.3}$$

Consequently $U\mu_n$ and $P_B U\mu_n$ are bounded for each n, and so using (2.2) and (3.1)

$$V\mu_D = \lim_n V\mu_n = \lim_n (U\mu_n - P_B U\mu_n) = \lim_n \sum_{k=0}^{n} (p_k - P_B p_k) = p .$$

Therefore

$$P^x(T_D < T_B) = p(x) = V\mu_D(x) ; \tag{3.4}$$

that is, μ_D as defined in (3.2) is the capacitary measure of D relative to the process (X, T_B).

Next suppose that $\sum p_k$ is bounded, or only finite, for each x. Conditions guaranteeing this are given in (2.4) and (2.10). Then from (3.3), $U\mu_D = \sum_{k\geq 0} p_k$ is finite and so (3.4) may be written

$$p = V\mu_D = U\mu_D - P_B U\mu_D = U\mu_D - U\hat{P}_B\mu_D = U(\mu_D - \hat{P}_B\mu_D)\,.$$

If we define $\nu = \mu_D - \hat{P}_B\mu_D$, then ν is a signed measure such that $U\nu(x) = p(x) = P^x(T_D < T_B)$. Therefore $U\nu = 1$ on D^r—the regular points of D—and 0 on B^r. But $\bar{D}$ and $\bar{B}$ are disjoint, and so $\nu^+ = \mu_D$ is carried by $\bar{D}$, more precisely by $D \cup {}^rD$ where ${}^rD$ is the set of coregular points of D, while $v^- = \hat{P}_B\mu_D$ is carried by $\bar{B}$, more precisely by $B \cup {}^rB$. In other words ν is the "condenser charge" for D and B and the formula

$$\nu = \mu_D - \hat{P}_B\mu_D \tag{3.5}$$

says that ν^+ is the capacitary measure μ_D of D relative to (X, T_B) and that ν^- is the balayage of $\nu^+ = \mu_D$ on B.

REMARKS. Of course, using the methods of Revuz [6], one can establish the existence of a measure μ_D such that $p = V\mu_D$ under duality and mild transience hypotheses. Then it is immediate that

$$U\mu_D = V\mu_D + P_B U\mu_D = p + U\hat{P}_B\mu_D\,. \tag{3.6}$$

But an additional "finiteness" argument seems to be necessary in order to conclude from (3.6) that

$$p = U\mu_D - U\hat{P}_B\mu_D = U(\mu_D - \hat{P}_B\mu_D)\,.$$

Our approach shows that whenever π_D exists, then μ_D exists and is given by (3.2).

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