CELEBRATIO MATHEMATICA

R. L. Moore

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While one cannot say that the "Moore Method" of teaching mathematics has gained wide-spread acceptance in college and university circles, it has been and is being successfully used by enough people to attract attention — even outside academia. Furthermore, there is considerable interest and curiosity about what it is and how it works. Mainly, for this reason, I am going to describe my own experiences in Moore's classes and in using the method with my own students, both graduate and undergraduate.

A bit of history

While he was still a graduate student at the University of Chicago (1903–05), R. L. Moore conceived the basic ideas that led eventually to his rather radical method of teaching. With his quick mind and restless spirit he found the lecture method rather boring — in fact, mind dulling. To liven up a lecture he would run a race with his professor by seeing if he could discover the proof of an announced theorem before the lecturer had finished his presentation. Quite frequently he won the race. But in any case, he felt that he was better off from having made the attempt. So if one could get students to prove theorems for themselves, not only would they have a deeper and longer lasting understanding, but somehow their ability and interest would be strengthened. If the theorems were too difficult, then they would have to be broken down into easier lemmas.

The more he thought about having the students discover for themselves the mainstream of a subject, the more he became convinced that it not only could be made to work but that it would also be attractive to students. He spoke of this plan to one of his professors (possibly Veblen) who abruptly replied: "Ha, let the students do the work!" But E. H. Moore's reaction was more thoughtful — perhaps the approach did have merit.

As a beginning instructor it was not easy for Moore to find a department where he had sufficient freedom to give the idea a really good try. But when he did, he began (at The University of Pennsylvania) to have success, especially in the Foundations of Geometry. Here was a fresh, relatively new area where Moore had himself tested the difficulty of some of the theorems. In the years following his appointment at The University of Texas, he expanded the use of his limited lecture method to all of his classes: Calculus, Advanced Calculus, Measure Theory, Metric Density, etc., as well as Foundations of Geometry and (at the graduate level) Topology. The results were, from the standpoint of research productivity after the Ph.D., really phenomenal. Of the Ph.D's produced in the U.S. and Canada during the period 1915–1954,

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25% of those from Texas were among the top 15% in the nation in productivity; 5% of those from U.C. (Berkeley) were in the top 15%; 8% of those from Chicago were in the top 15%; 16% of those from Harvard were in the top 15%; 20% of those from Princeton were in the top 15%. (A Survey of Research Potential and Training in the Mathematical Sciences, The University of Chicago, March 15, 1957; *The Albert Report*.) And it should be no surprise that high quality was associated with high productivity. Included in this group from Texas are Moore's students R. L. Wilder, G. T. Whyburn, J. H. Roberts, R. H. Bing, E. E. Moise, R. D. Anderson, and Mary Ellen Rudin (in chronological order).

What Moore did

Moore would begin his graduate course in topology by carefully selecting the members of the class. If a student had already studied topology elsewhere or had read too much, he would exclude him (in some cases, he would run a separate class for such students). The idea was to have a class as homogeneously ignorant (topologically) as possible. He would usually caution the group not to read topology but simply to use their own ability. Plainly he wanted the competition to be as fair as possible, for competition was one of the driving forces. (For the foundations of geometry he made no attempt to select the students because all of them, young and old, high school teachers or not, were uniformly ignorant of the Hilbert–Veblen–Moore axiomatic approach to the subject.)

Having selected the class he would tell them briefly his view of the axiomatic method: there were certain undefined terms (e.g., "point" and "region") which had meaning restricted (or controlled) by the axioms (e.g., a region is a point set). He would then state the axioms that the class was to start with (Axioms 0 and 1 of his book: Foundations of Point Set Theory, omitting part (4) of Axiom 1). An example or two of situations where the axioms could be said to apply (e.g., the plane or Hilbert space) would be given. He would sometimes give a different definition of region for a familiar space (e.g., Euclidean 3-space) to give some intuitive feeling for the meaning of an "undefined term" in the axiomatic system. Of course, this was part of his own personal philosophy and he considered it part of the motivation of the subject.

After stating the axioms and giving motivating examples to illustrate their meaning he would then state some definitions and theorems. He simply read them from his book as the students copied them down. He would then instruct the class to find proofs of their own and also to construct examples to show that the hypotheses of the theorems could not be weakened, omitted, or partially omitted.

When the class returned for the next meeting he would call on some student to prove Theorem 1. After he became familiar with the abilities of the class members, he would call on them in reverse order and in this way give the more unsuccessful students first chance when they *did* get a proof. He wasn't inflexible in this procedure but it was clear that he preferred it.

When a student stated that he could prove Theorem x, he was asked to go to the blackboard and present his proof. The other students, especially those who hadn't been able to discover a proof, would make sure that the proof presented was correct and convincing. Moore sternly prevented heckling. This was seldom necessary because the whole atmosphere was one of a serious community effort to understand the argument.

When a flaw appeared in a "proof" everyone would patiently wait for the student at the board to "patch it up." If he could not, he would sit down. Moore would then ask the next student to try or if he thought the difficulty encountered was sufficiently interesting, he would save that theorem until next time and go on to the next unproved theorem (starting again at the bottom of the class).

Occasionally theorems got left over indefinitely but nearly all of these would be proved in some subsequent year.

Quite frequently when a flaw would appear in a proof everyone would spend some time (possibly in class) trying to get an example to show that it couldn't be "patched up," i.e., a counterexample to the argument (even though the theorem might be correct). This kind of experience is seldom encountered in courses or in any place outside of one's own research work. Yet this kind of activity is vitally necessary for the research worker.

Occasionally an improvement on one of Moore's theorems would be proved. Moore would then refer to that theorem with the student's name. (In the revision of his book a number of names like this appear in the text and in some cases he made remarks concerning the origin of certain proofs and concepts in the appendix.) The improvement might not be very significant but the encouragement given by the "public recognition" was considerable. At the same time, mistakes were not discouraged. However, on certain theorems (the arcwise connectivity theorem, for example) knowing from experience that mistakes were likely, Moore would insist that the student have the proof written out before presenting it. This tended to reduce the number of false starts (and also gave the student some useful writing practice.)

Difficulties and drawbacks

Probably the most obvious difficulty is the one of class uniformity. If the competition is not reasonably on an equal footing, then one of the basic drives has been weakened. Obviously, in contrast to the 1920s, little can be done about this now when almost all undergraduates have had some exposure to topology and topological ideas. G. T. Whyburn spoke to me more than once about how to handle this problem. My solution (or partial solution) is to begin general topology with only the most basic axiom: there exists a non-empty set S of elements called points and a collection G of subsets of S such that G covers S. Calling the elements of G "regions" and using the usual definitions of "limit point," "closed," and "open," some of the usual propositions about these notions are false but when carelessly formulated the usual arguments seem to work. This forces a kind of uniformity on the class. (I continue this introductory course by supposing that the space is semi-metric, i.e., metric without the triangle inequality, and everything returns to normal — derived sets are closed, the union and intersection of closed sets are closed, etc.) However, I do not find the lack of uniformity to be a severe handicap because the class becomes more homogenous as to background as time goes on.

There is a problem of what to do with students who are too timid to present their proofs at the board. I generally try to draw them into the discussions, offer to do the writing on the board for them while they stay seated, and eventually after six months or so they get up without realizing what they are doing — especially if a subtle argument gets rather heated. But when there are students who don't present theorems or counterexamples in class, I simply depend upon the final examination to determine the course grade. For those not actively participating, the course becomes a lecture course (the lectures lacking somewhat in polish).

There is also the student whose proof gets "shot down." People generally tend to be embarrassed by mistakes — especially public mistakes — and care must be taken not to make them feel that criticism is ridicule. As working mathematicians we have become so accustomed to making mistakes that we are inclined to forget the pain experienced by the young.

Then there is the problem of *reading*. After a few months of working out their own proofs it becomes quite difficult to get students to read mathematics. They would rather do it themselves. Nevertheless, reading is necessary if one is to become educated (especially in the brief span of graduate school). Moore used to complain that when he wanted (finally) the students to read, they couldn't. After some trial and error I have found the following technique to be effective. Have the students in the beginning graduate general topology not read until Christmas and then buy themselves a copy of Kelley. They can then use Kelley for bedtime reading to prepare for the final examination (and the qualifying examination — I make them the same) which will be mostly (70%) on Kelley. I never discuss Kelley in class. The students are supposed to do this reading unassisted. Class continues as before with the students working out their own proofs for by then we are going in a direction which overlaps Kelley very little.

By far the most difficult aspect of the method is patience. The instructor must not help — must not point out the "obvious." We all know how difficult even the obvious is before it becomes obvious. The instructor must simply be willing to wait for the students' mental chemistry to work. It helps if the instructor feels rewarded when the student *does* finally see how to put together a few ideas correctly.

And finally there is the problem of what to do when "no one has anything." One can, of course, start a new topic, motivate it, and get the students to construct some examples to fit, or state some new definitions and theorems. But this kind of thing cannot be done very often for there must not be too many unsettled problems or the student will become distracted and unable to concentrate his attack. I have several topics which can become useful side issues: (1) problems about set theory, well-ordering and cardinality can be taken up and worked out by the students on the spur of the moment, (2) how some of the theory simplifies if one assumes the space to be metric, (3) the history of some of the ideas and the personalities of some of the people involved, etc. But generally I make up some questions involving the application of theorems already proved (or even those yet to be proved) which can be settled in ten or fifteen minutes each.

At such times one can introduce the beginnings of notions, usually in connection with examples, that will be useful on theorems to come. That is, one can somewhat randomly (and out of context) put into the subconscious minds of the students pictures, ideas and notions which will resurface weeks (or even months) later in a proof. There should be no hint as to what the ideas are really for and in fact the student when he later uses one of the notions will have the feeling that he discovered it himself. In this way, as well as the actual statement of applicable lemmas, the proofs of quite difficult theorems can be made accessible.

Where can the method be used?

One frequently hears people say: "Yes, it may work in topology but not in ..." I will discuss briefly certain areas where I have found it to work very well.

One of the most rewarding is group theory. Here again one may begin with rather simple axioms and the central theory can be broken down into a rather nice sequence of interesting but easy theorems. With two different classes I have used Speiser's *Die Theorie der Gruppen* with modifications (numerous illustrative examples should be presented by both the students and the instructor and Theorem 40 (Frobenius) is too difficult without the introduction of a lemma).

As previously mentioned, the axiomatic development of the Hilbert–Veblen–Moore Foundations of (plane) Geometry works extremely well. There are proofs in this subject for students of widely varying abilities (the "mid-point" theorem: If ab is an interval, it contains a point c such that ac is congruent to cb, is too difficult without the Dedekind Cut Axiom).

An introduction to Hilbert Space works very well from von Neumann's 1928 paper (see M. H. Stone's colloquium publication — some of the lemmas may be omitted and left to the students to discover). The start of this theory makes a nice six to ten week course. I once told von Neumann about doing this and he remarked that it would require a very good class. The class was good and being a summer course, the students were already experienced in "proving things for themselves." However, once the ideas are stated and the sequence of theorems laid out, the rest is much, much easier than the situation von Neumann faced.

These are areas where a set of axioms can be used for a beginning. What about other areas? I have personally used the method in courses in real variable theory and in complex variable theory. H. S. Wall and some of his students, MacNerney and Porcelli, in particular, have used the method (with obvious success) in various courses of analysis. Some areas require quite some effort in formulating a good sequence of theorems. MacNerney worked two years to develop a sequence (in Complex Variable Theory) which would yield the Cauchy Integral Theorem in one quarter (one semester is better).

For the topology of the line (and the plane) I have found the non-axiomatic approach to be more successful (for beginning college and high school students). I spend some time being certain that the class is clear on the properties of the natural numbers including order, well-order and induction. A "point" is defined to be a function from the natural numbers to the natural numbers. Then giving the set of all such functions the lexicographic order and using the open interval topology (except at the constant function 1 where the correct half-open interval is necessary), one has an interesting beginning of a development that quickly gives all the topological properties of the non-negative real numbers. The "sum" and "product" of points given by the usual definitions for functions is discontinuous. Hence the discovery of a perfect set which contains no interval presents the student with something like the situation that Cantor encountered.

Of course, there is also a simple axiomatic approach to the topology of the line (which omits arithmetic) and Burgess has used it quite successfully.

Moore often gave a summer course in metric density in the plane and the line where the beginning was non-axiomatic. In fact, a cursory knowledge of Lebesgue measure was required.

Who can use the method?

I have already mentioned that the instructor must possess "patience." But I think it is more than that. It must be "patience" that is born of the conviction that training a student to do research is important — even more important than conveying knowledge; that trying to develop a student's mathematical ability to the limit of that ability is important.

The instructor should suppress his own urge to get into the act. Even an ugly proof from one of the students should please the instructor. Only on rare occasions (possibly "never" is the better policy) should he show an elegant proof. The student can learn "elegance" from his reading later. Moore once did this to me and his elegant argument drove my ugly one out of my mind and I have wished many times later that I could recall it where Moore's technique would not work. Maybe my ugly one would have. this is not to say that wording and clarity of thought should not be discussed. But it should be done on the student's own argument.

The instructor should work out the sequence of theorems keeping in mind the general needs and abilities of a group as the course proceeds so that extra lemmas may be introduced (for a weak group) or some lemmas may be omitted (for a strong group). I even like to leave the general direction of the course flexible so as to accommodate the interests of the group as I become better acquainted with the various individuals. The main thing, as often expressed by W. M. Whyburn, is to "give them something they can do."

Wilder has expressed the opinion that Moore was successful in using the method because the students were proving Moore's theorems while the development of the theory was still hot. I expect this may have been a factor in the obvious enthusiasm in the Moore School of the 1920s. This was absent later (say in the 1950s or a bit earlier) and Moore was just as successful. Clearly to be successful, the instructor should give the student puzzle after puzzle that the student cannot resist. His appetite (and ability) increases with every solution.

Techniques

In teaching General Topology several people have followed Moore's technique of simply assigning the class the sequence of theorems as they occur in Moore's book. Most people, however, have used some variant of the method more suitable to their own ideas of what develops the student. For example, Bing has some student in the class act as secretary and write up the notes for the whole class. By passing this chore around, some of the students get some practice at writing. (And it's a good idea for other reasons.) I like to state false propositions (just as if they were true) for the students to prove. And quite frequently I state a number of definitions and ask the students to formulate some theorems using them. I feel that examples (and counterexamples) are very important for both understanding and motivation. In particular, one's intuition is aided by examples of spaces that do not satisfy the axioms as much as by examples that do: for instance, (in my approach to General Topology) topological spaces (even compact and Hausdorff) that are not semi-metric and semi-metric spaces which are not metric.

And it's nice to have a sequence of theorems which are useful but which can be proved by anybody. Elementary properties of connected point sets can be formulated into a sequence of this sort.

It is a good plan to encourage students to change a theorem until they can prove it: weaken the conclusion or strengthen the hypothesis or both. This helps to avoid frustration and is good practice.

Final remarks

The instructor should convey confidence, especially in the beginning. The student should soon learn that some things he can do quickly but others may take effort and time. Given six months practice, a student who had never thought of a proof in his life and didn't know how to start, may develop to the point where he can settle almost anything you propose. I find this very rewarding and satisfying. It happens often enough to keep one's enthusiasm for teaching vitally alive.

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