**Topic:** The space of Casson homomorphisms for a surface $K_j$.

**The idea:**

- $M =$ moduli class group of $K_j$.
- $H = H_1(K_j, \mathbb{Z})$.
- $J = \text{Torelli}(K_j)$.
- $J$ is the subgroup of $\text{Aut}(K_j)$ consisting of automorphisms that act trivially on the Jacobian variety $J(K_j)$.

Then there is a family of homomorphisms $\lambda_L : J \to \mathbb{Z}$ defined by Casson's invariant $\lambda_L$.

The parameter $L$ ranges over the linking form on $K_j$, that is, a bilinear $L : H \otimes H \to \mathbb{Z}$ such that $L(x, y) = -L(y, x)$ for all $x, y$.

$L - LT = J$.

We will find the structure and rank of the abelian group $\Lambda$ of homomorphisms $J \to \mathbb{Z}$ generated by the $\lambda_L$. Although the $\lambda_L$ are infinite in $\Lambda$, $\Lambda$ is free abelian of finite rank $1 + \frac{g^2(g^2-1)}{2}$.

Also, $\Lambda$ is homomorphic to $\mathbb{Z}^3$ a certain $GL$-irreducible tensor construction on $H$ which is equivalent to the Riemann curvature tensor.
I. The Casson Hom $\lambda_k$ for $K=K_2$ a Surface $S^3$

1. For $K=K_2 \subset \mathbb{R}^3$ homology sphere and $f$ a diffeo of $K$,
form a 3-manifold $Y^3(K,f)$ by:

A. Slice $S^3$ along $K$.

B. Reglue $K$ to $K'$ by identifying
$x$ in $K$ with $f(x)$ in $K'$.

Note: if $f=T_\phi$, where $\phi$ is a BSSC on $K$ then $Y(K,T_\phi)$
is the homology sphere got by doing $\nu_1(\phi)$ surgery on $Y_{\mathbb{C}^3}$.
Also, by Casson, $Y(K,T_\phi) = \chi(\emptyset < S^3)$.

For $f \in \mathcal{F}(K)$, define $\lambda_k(f)$ to be $\lambda(Y(K,T_\phi))$.

2. If $f = \prod_{i=1}^m T_{\phi_i}$, then $\lambda_k(f) = \sum_{i=1}^m \lambda'(\emptyset < S^3)$

Proof: Let $n=2$ and the rest will be obvious.

Gluing at $T_{\phi_1}, T_{\phi_2}$ is the same as doing a surgery
on $\partial_2 < K$ and another on $\partial_1 < K$ got by pushing
$K$ up in the positive direction.

The gluing done is by $T_{\phi_2}$ as we pass upward through $K$ and $T_{\phi_1}$ as we pass through $K'$

Doing the first surgery, we get
$\lambda_k(T_{\phi_1}) = \lambda(Y(K,T_{\phi_2})) = \chi(\emptyset < S^3)$

Doing the second surgery, we get
$\lambda_k(T_\phi) = \lambda(Y(K,T_{\phi_1}, T_{\phi_2}) = \chi(\emptyset < Y(K,T_{\phi_2})) + \chi(Y(K,T_\phi))$

$= \chi(\emptyset < Y(K,T_\phi)) + \chi'((\emptyset_1 < S^3) + \chi''((\emptyset_2 < S^3))$

By Casson
$= \chi(\emptyset < S^3) + \chi''((\emptyset_1 < S^3) + \chi''((\emptyset_2 < S^3)$
But \( \gamma_1^+ \cup \gamma_2^- \) is a boundary link, since \( \gamma_1^+ \cup \gamma_2^- \) bound disjoint
surfaces in \( K^+ \cup K^- \). So \( \lambda_v = 0 \). For any

\[ \lambda_K(T_0 T_2) = \lambda(K_1^+) + \lambda(K_2^-) = \lambda(K_1^+) + \lambda(K_2^-) = \]

\[ = \lambda_K(T_0^+) + \lambda_K(T_0^+) \quad \equiv \quad \text{QED} \]

This shows that \( \lambda_K(fg) = \lambda_K(f) + \lambda_K(g) \) for \( f, g \in \mathcal{F} \),
so \( \lambda_K \) is a hom. \( \mathcal{F} \to \mathbb{R} \).

3. To see how \( \lambda_K \) behaves on \( \mathcal{F} \), we will compute it on generators
for \( \mathcal{A} \). \( \mathcal{A} \) is generated by \( B_1 \), \( C_1 \), \( C_1^{-1} \). But we can do better.
\( \mathcal{A} \) is seen by \( B_1 \), \( C_1 \), \( C_1^{-1} \) alone which bound

A SUBSURFACE of \( K \) of genus 1 or 2.

Proof: We start with the relation on a quartic with 3 holes (two 2): \( T_0 T_3 T_3 = T_0 T_1 T_2 \) (mod 2).

Then \( T_0 T_1 T_2 T_3 T_4 \) (mod 2).

Now view picture to

\[ \lambda_2 \]

\[ \lambda_3 \]

\[ \lambda_1 \]

\[ \lambda_4 \]

Now view picture to set \( \lambda \), a loop of \( \mathcal{F} \)'s \( T_0^+ \)'s, \( T_1^+ \)'s, \( T_1^+ \)'s.

\[ \text{QED} \]

So we need only calculate \( \lambda_K \) on \( \mathcal{F} \) of genus 1 or 2, respectively.

4. To calculate \( \lambda_K(T_0) \) we need the Alexander polynomial of \( \gamma \), which
we obtained from the linking form of \( \gamma \) or a \( \gamma \) self-surface for \( \gamma \). Since \( \gamma \) bounds
in \( K \), we have a (unique) subsurface of \( K \) as a self-surface for \( \gamma \), and

can use the linking form \( L \) of \( K \) on \( S^2 \), restricted to \( \gamma \). We have

two now for genus 1 or 2, curves \( \gamma \).

**Lemma 1:**
\[ L \begin{pmatrix} L(x, y) \\ L(y, x) \end{pmatrix} = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \]

Alexander = \( a + b + c = \text{det} (tL - L^T) \) where \( a = \text{det} L \)

Sym Alexander = \( a + b + c \) \( = \frac{1}{2} D^2 \) (Alexander) \( = 2(2a) = a = \text{det} L = \begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} \)

**Lemma 2:**
\[ L = \begin{pmatrix} L_{ij} \end{pmatrix} \]

Alexander = \( \text{det}(tL - L^T) = a^4 + b^2 + c^2 + d^2 + g \)

Sym Alexander = \( a^2 + b^2 + c^2 + d^2 + g \) \( = \frac{1}{2} D^2 \) (Sym Alexander) \( = 2a + b \)

\[ a = \text{det} L \quad b = \sum_i \text{det} (\text{column } i \text{ of } L \text{ from } \text{column } i \text{ of } L) \]

So \( 4a + b = \sum_i \text{det} (\text{column } i \text{ of } L \text{ from } \text{column } i \text{ of } L) \)

Since \( L - L^T = J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) each of the above is evaluated early at a 3x3 det.

\[
\begin{vmatrix} L_{12} & L_{13} & L_{14} \\ L_{23} & L_{24} & L_{24} \\ L_{34} & L_{34} & L_{34} \end{vmatrix} = \begin{vmatrix} L_{11} & L_{13} & L_{14} \\ L_{13} & L_{13} & L_{14} \\ L_{14} & L_{14} & L_{14} \end{vmatrix} + \begin{vmatrix} L_{11} & L_{13} & L_{14} \\ L_{13} & L_{13} & L_{14} \\ L_{14} & L_{14} & L_{14} \end{vmatrix} + \begin{vmatrix} L_{11} & L_{13} & L_{14} \\ L_{13} & L_{13} & L_{14} \\ L_{14} & L_{14} & L_{14} \end{vmatrix} + \begin{vmatrix} L_{11} & L_{13} & L_{14} \\ L_{13} & L_{13} & L_{14} \\ L_{14} & L_{14} & L_{14} \end{vmatrix} + \begin{vmatrix} L_{11} & L_{13} & L_{14} \\ L_{13} & L_{13} & L_{14} \\ L_{14} & L_{14} & L_{14} \end{vmatrix}
\]

**Example:**
\[
\begin{vmatrix} L_{12} & L_{13} & L_{14} \\ L_{23} & L_{24} & L_{24} \\ L_{34} & L_{34} & L_{34} \end{vmatrix} = \begin{vmatrix} L_{11} & L_{13} & L_{14} \\ L_{13} & L_{13} & L_{14} \\ L_{14} & L_{14} & L_{14} \end{vmatrix} + \begin{vmatrix} L_{11} & L_{13} & L_{14} \\ L_{13} & L_{13} & L_{14} \\ L_{14} & L_{14} & L_{14} \end{vmatrix} + \begin{vmatrix} L_{11} & L_{13} & L_{14} \\ L_{13} & L_{13} & L_{14} \\ L_{14} & L_{14} & L_{14} \end{vmatrix} + \begin{vmatrix} L_{11} & L_{13} & L_{14} \\ L_{13} & L_{13} & L_{14} \\ L_{14} & L_{14} & L_{14} \end{vmatrix} + \begin{vmatrix} L_{11} & L_{13} & L_{14} \\ L_{13} & L_{13} & L_{14} \\ L_{14} & L_{14} & L_{14} \end{vmatrix}
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\]

**Examples of Top Row:**
\[
L_{21} \text{ (Downwards Change)}
\]

\[-L_{13} \begin{pmatrix} L_{23} & L_{24} \\ L_{23} & L_{24} \end{pmatrix} - L_{13} \begin{pmatrix} L_{23} & L_{24} \\ L_{23} & L_{24} \end{pmatrix} = -L_{13} \begin{pmatrix} L_{23} & L_{24} \\ L_{23} & L_{24} \end{pmatrix} = L_{13} L_{24} \]

\[L_{12} \begin{pmatrix} L_{23} & L_{24} \\ L_{23} & L_{24} \end{pmatrix} - L_{12} \begin{pmatrix} L_{23} & L_{24} \\ L_{23} & L_{24} \end{pmatrix} = L_{12} \begin{pmatrix} L_{23} & L_{24} \\ L_{23} & L_{24} \end{pmatrix} = -L_{12} L_{23} \]

\[L_{13} \begin{pmatrix} L_{24} & L_{24} \\ L_{24} & L_{24} \end{pmatrix} - L_{13} \begin{pmatrix} L_{24} & L_{24} \\ L_{24} & L_{24} \end{pmatrix} = L_{13} \begin{pmatrix} L_{24} & L_{24} \\ L_{24} & L_{24} \end{pmatrix} = -L_{13} L_{24} \]

\[L_{14} \begin{pmatrix} L_{24} & L_{24} \\ L_{24} & L_{24} \end{pmatrix} - L_{14} \begin{pmatrix} L_{24} & L_{24} \\ L_{24} & L_{24} \end{pmatrix} = L_{14} \begin{pmatrix} L_{24} & L_{24} \\ L_{24} & L_{24} \end{pmatrix} = -L_{14} L_{24} \]

So \( 2 \sum_{i=1}^{14} \text{ det } \begin{pmatrix} L_{i1} & L_{i2} \\ L_{i2} & L_{i2} \end{pmatrix} \text{ (Sum of Elements) } \)
Now let $K$ be an abstract surface of type $9,1$ (not immerse in $S^3$).

For any embedding $\phi$ of $K$ into $S^3$, we get a Casson homomorphism $\lambda_\phi : \mathcal{J}(K) \to \mathbb{Z}$ which is induced from $\phi(K) \subset S^3$; that is, for $f \in \mathcal{J}(K)$, $\phi f \phi^{-1}$ is in $\mathcal{J}(\phi(K))$ so we can define $\lambda_\phi(f) = \lambda_{\phi(K)}(\phi f \phi^{-1})$. This has the following consequences:

A) If $h \in \mathcal{M}(K)$ then $\phi h$ is also an embedding of $K \subset S^3$, and we can define

$$\lambda_{\phi h}(f) = \lambda_{\phi(K)}(\phi h f h^{-1} \phi^{-1}) = \lambda_\phi(h f h^{-1})$$

B) The value of $\lambda_\phi(f)$ is calculated by using the linking form on $\phi(K)$, restricted to the surface $\phi(S)$ bounding $\phi(K)$.

Pulling this form back via $\phi$ to $K$ gives us an induced form $L_\phi$ on $K$ from which we can also compute the linking information:

$$L_\phi = \lambda_{\phi}(T_{\phi}) = \text{the value got by applying the } L\text{-formula to } S \text{ using } L_\phi.$$  

Note: $L_\phi$ is a linking form on $K$. $L_\phi$ is $L$.

C) By the above, $\lambda_\phi(f)$ depends only on $L_\phi$, not on $\phi$ itself. For this reason, we can write $\lambda_L(f)$ instead of $\lambda_{\phi}(f)$.

Note: Given any linking form on $K$, there is an embedding $\phi : K \subset S^3$ inducing this form; hence, for every linking form $L$ there is a unique Casson homomorphism $\lambda_L : \mathcal{J}(K) \to \mathbb{Z}$. 
D) \( \lambda_L(h f h^{-1}) = \lambda_{L'}(h f h^{-1}) = \lambda_{h L}(f) = \lambda_{L h}(f) \) for each matrix \( h \in G \) and each \( f \in \mathbb{K} \). Note also that the action of \( h \) on \( L \) occurs through \( S_h \) — that is, \( S_h \) is an \( S \)-equivariant map on the Casson norm.

E) If \( h \in L \), then \( h^+ L = L \), so get:

\[ \lambda_L(h f h^{-1}) = \lambda_L(f) \text{ for } h \in L, \text{ all } L \]

In particular:

\[ \lambda_L[\mathcal{L}, \mathcal{F}] = 0 \text{ for all } L \]

We have been looking at \( \lambda_L(f) \) as a map \( L \to \mathbb{Z} \), i.e., as a function of \( f \). To get some global info about the space \( \mathcal{L} = \text{Hom}(\mathcal{F}, G) \) of Casson norms, we now take an additional point of view: 

We will fix \( \gamma \) to be a curve of genus 1 on \( \Sigma \) and think of \( \lambda_L(T \gamma) \) as a function of the linking form \( L \).

To begin: let \( \mathcal{L} = \text{the space of linking forms} \) \( L \) on \( K \):

\[ L - L^* \to J \]

The first thing to notice about \( \mathcal{L} \) is that it is a linear space of all bilinear forms \( h^* \otimes h^* \) on \( H \); but it is in fact an affine linear surface of \( h^* \otimes h^* \) i.e., it does not pass through the origin of \( H^* \otimes H^* \).

In fact we have:

A) If \( L_1, L_2 \) are two linking forms, then \( L_1 - L_2 \) is symmetric.

\[ J T = J \] on \( S(\mathcal{X}, \mathcal{Y}) = S(\mathcal{Y}, \mathcal{X}) \)

B) Conversely, if \( S \) is symmetric, and \( L \) is a linking form, then \( L + S \) is a linking form.
c) Hence \( \mathcal{L} \) is just a translate of the symmetric bilinear forms \( \wedge^2(\Lambda^k) = \Lambda^k \otimes \Lambda^k \).

(6) If \( f \in \mathcal{L} \), we now write \( \lambda_f : \mathcal{L} \to \mathbb{Z} \), defined by

\[
\lambda_f(L) = \lambda_L(f) : \text{\( \lambda \) is the "adjoint" of \( \lambda \).}
\]

Note that

\[
\lambda_{f_1 + f_2}(L) = \lambda_{f_1}(f_2) = \lambda_{f_2}(f_1) = \lambda_L(f_1) + \lambda_L(f_2) = \lambda_{f_1}(f_2).
\]

Any such \( \lambda \) has a homomorphism from \( \mathcal{L} \) to the vector space of functions \( \mathcal{L} \to \mathbb{Z} \).

Looking at the \( L \)-formula for groups 1 and 2 converge, we see that \( \lambda(\mathbf{g}) \) is actually an (affine) quadratic function on \( \mathcal{L} \) (as the formula are quadratic in the correct \( \mathbf{g} \)).

Hence \( \lambda \) already maps all of \( \mathcal{L} \) into quadratic functions on \( \mathcal{L} \to \mathbb{Z} \).

\[
\lambda(\mathcal{L}) = \text{affine quadratic functions on } \mathcal{L} \to \mathbb{Z}
\]

ie. \( \lambda : \mathcal{L} \to \text{Quad}(\mathcal{L}, \mathbb{Z}) \)

Note that \( \text{Quad}(\mathcal{L}, \mathbb{Z}) \) is a free abelian group of finite rank.

(9) We now define \( \mathbb{C} \) to be the common kernel of all the \( \lambda_s \).

\[
\mathbb{C} = \cap \ker \lambda_s \quad \text{and we put} \quad \mathcal{T} = \frac{\mathcal{L}}{\mathbb{C}}, \quad \text{which makes sense}
\]

Since \( \mathbb{C} \) is certainly normal in \( \mathcal{L} \), in fact \( \mathbb{C} \) contains \( \mathcal{T} \) since each \( \ker \lambda_s \) does, so \( \mathcal{T} \) is abelian, and every \( \lambda_s \) factors through \( \mathcal{T} \), and hence also every \( \lambda_s \) in \( \wedge \mathcal{T} \cdot \text{Hom}(\mathcal{T}, \mathbb{Z}) \). We thus have

\[
\wedge \subset \text{Hom}(\mathcal{T}, \mathbb{Z}) = \mathcal{T}^\ast
\]
Then we have the following:

A) \( \sigma \) is the zero function \( \mathcal{L} \) iff \( q_\ell(t) = 0 \) for all \( t \), i.e., iff \( f \in \mathcal{L} \). Thus \( \sigma : \mathcal{L} \to \text{Quad} ( \mathbb{Z}^{2} ) \)

are fractions through \( \tau \) and \( \tau : \mathcal{L} \to \text{Quad} ( \mathbb{Z}^{2} ) \) is \( \Gamma \).

B) \( \tau \) implies that \( \mathcal{L} \) is finite rank free abelian, since \( \text{Quad} ( \mathbb{Z}^{2} ) \) is so. Hence \( \lambda \Lambda \mathcal{L}^* \) is also finite rank free abelian.

C) Suppose \( \text{rank} \Lambda < \text{rank} \mathcal{L}^* \); then there would exist a \( t \in \mathcal{L} \), \( t \neq 0 \), such that \( \chi(t) = 0 \) for all \( \lambda \in \Lambda \); but we have already seen that \( \chi(t) = 0 \) for \( \Lambda \Rightarrow t = 0 \in \mathcal{L} \).

Hence: \( \text{rank} \Lambda = \text{rank} \mathcal{L}^* = \text{rank} \mathcal{T} \)

and \( \Lambda \) is finite index in \( \mathcal{T}^* \).

(0) Since \( \Lambda \) is virtually dual to \( \mathcal{T} \), we will investigate the \( Z \)-values via \( \sigma \) with a subgroup of the quadratic functions on the affine space \( \mathcal{L} \).

Also, we now abandon our restriction to \( Z \)-values and tensor everything with \( Q \); this will smooth out certain messy problems with splitting, etc.

Thus \( \mathcal{T} \) is now contained as a subspace of \( \text{Quad} ( \mathbb{Z}^{2}, Q_{\mathbb{Z}} ) \).

(1) To begin, we need to understand the affine linear functions \( \text{Lin} ( \mathcal{L}^{*}, \mathbb{Z} ) \), which we write as \( \mathcal{L}^{*} \) for short. Note that the space of affine linear functions is an affine space of dimension \( n + 1 \) with dimension \( n + 1 \), and includes the constant functions. There is no way to define them nontrivially.

Linear functions in this setting (2) given in the perfect setting, there is (3) nontrivial (4) linear functions on \( \mathcal{L} \), and hence \( \text{Lin} ( \mathcal{L}^{*}, \mathbb{Z} ) \) gives us in the obvious way linear functions on \( \mathcal{L} \).
\[(x \otimes y)(L) = L(x \otimes y)\]

Let \(\mathcal{H}^2\) be defined as the function generated by
\[x \otimes y = \frac{x \otimes y + y \otimes x}{2}\]
and \(\mathcal{H}^2\) obtained by gluing by

The monomials \(x \otimes y\) are such that \(\mathcal{H}^2\) splits
as \(\mathcal{H}^2 \oplus \mathcal{H}^2\)
and this splitting is natural under the action of \(G\) on \(\mathcal{H}^2\).

Also, we have the \(J\) map \(J: \mathcal{H}^2 \rightarrow \mathcal{Q}\) given by
\[x \otimes y \mapsto x \otimes y\]
for all \(x, y \in \mathcal{H}^2\).

Let \(\theta \in \text{ker} J\) and \(\theta \in \mathcal{H}^2\), then \(\theta(L) = 0\) for all \(L\).

If \(\theta \in \text{ker} J\), then \(\theta\) is a constant function on \(L\).

Proof: \(\theta = \frac{1}{2} \sum x \otimes y \in J\) iff \(\sum x_i \otimes y_i = 0\)

Hence, \(\theta(L) = \frac{1}{2} \sum (x \otimes y)\) is constant on \(L\), and \(\theta = 0\) if \(\theta\) is in \(\text{ker} J\).

Conversely, \(\frac{\mathcal{H}^2}{\text{ker} J}\) act on linear functions on \(L\).

Now \(\frac{\mathcal{H}^2}{\text{ker} J} = \frac{\mathcal{H}^2}{\text{ker} J} = \frac{\mathcal{H}^2}{\text{ker} J} = \frac{\mathcal{H}^2}{\text{ker} J}\) since:

a) \(\text{dim} \mathcal{H}^2 = \text{dim} \frac{\mathcal{H}^2}{\text{ker} J}\) so \(\dim \mathcal{H}^2 = \dim \frac{\mathcal{H}^2}{\text{ker} J}

b) All linear functions on \(L\) are represented in \(\frac{\mathcal{H}^2}{\text{ker} J}\).

We have \(\frac{\mathcal{H}^2}{\text{ker} J}\) is the space \(\text{Lin}(L, \mathcal{Q}) = \mathcal{L}^\ast\)

Note that, \(\text{ker} J\) is the space of functions, the elements of \(\mathcal{H}^2\) such that \(\mathcal{H}^2\) are just the constant functions.

If \(x, y \in \mathcal{H}^2\), we let \([x \otimes y]\) represent the image of \(x \otimes y\)

in \(\frac{\mathcal{H}^2}{\text{ker} J}\); \([x \otimes y]\) is a linear function on \(L\) and \([x \otimes y](L) = L(x \otimes y)\).

Since \(\frac{\mathcal{H}^2}{\text{ker} J} = \mathcal{H}^2\), we can write \([x \otimes y]\) as:
\[ [xy] = \ln (x \circ y) = \frac{1}{2} \ln (x'y + x\circ y) = \frac{x \circ y + x \circ y}{\ln} \]

We are interested in the quadratic function space on \( \mathcal{F} \), i.e.,
\[ \mathcal{Q}(\mathcal{F}) = \mathcal{F}^2 (\mathcal{F}^2 \circ \mathcal{F}) = \mathcal{F}^2 (\mathcal{F}^2 \circ \mathcal{F}) \]

AND in particular, the image \( \mathcal{Q}(\mathcal{F}) \) of the map \( \mathcal{F} \to \mathcal{F}^2 (\mathcal{F}^2 \circ \mathcal{F}) \) which we have identified with \( \mathcal{F}^2 \). To determine the image.

More precisely, we look at the image of the group \( \mathcal{F} \) generated by two generators.

For a genus 2 generator \( \gamma \) (bending surface with homology basis \( a, b \)), we have
\[ \chi_2 (T_{xy}) = \frac{\text{L}(a, a) \text{L}(a, b)}{\text{L}(b, a) \text{L}(a, b)} = \text{L}(a, a) \text{L}(a, b) - \text{L}(a, b) \text{L}(b, a) \]

\[ = (\begin{bmatrix} a \end{bmatrix} \begin{bmatrix} b \end{bmatrix} - \begin{bmatrix} a \end{bmatrix} \begin{bmatrix} b \end{bmatrix}) \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \]

The function \( \mathcal{Q}(\mathcal{F}) \) on \( \mathcal{F} \) is given by the quadratic function
\[ \begin{bmatrix} a \end{bmatrix} \begin{bmatrix} b \end{bmatrix} - \begin{bmatrix} a \end{bmatrix} \begin{bmatrix} b \end{bmatrix} \]

But as a function, we have
\[ [xy] = x \circ y \quad x \circ y \]
\[ \begin{bmatrix} a \end{bmatrix} = a \circ a \quad \begin{bmatrix} b \end{bmatrix} = b \circ b \]
\[ \begin{bmatrix} ab \end{bmatrix} = a \circ b + 1 \quad \begin{bmatrix} ba \end{bmatrix} = b \circ a + 1 = a \circ b - 1 \]

Likewise, for a genus 2 generator \( \gamma_{2,2} \), we have
\[ \mathcal{Q}(\mathcal{F}) = \left( a \circ a \right) (b \circ b) - (a \circ b)^2 + 1 \]

Likewise, for a genus 2 generator \( \gamma_{2,2} \), we have
\[ \mathcal{Q}(\mathcal{F}) = \left( a \circ a \right) (b \circ b) - (a \circ b)^2 + 1 \]
The above formulas show that there is “no Lindelöf point” to \( \sigma(T_{x_1}) \), \( \sigma(T_{x_2}) \) i.e. they land in \( S^2(S^2 \mathbb{N}) \oplus \mathbb{Q} \) and induce

\[
\sum T = \lim_{n \to \infty} \sigma = S^2(S^2 \mathbb{N}) \oplus \mathbb{Q}
\]

We now must investigate the \( S^2(S^2 \mathbb{N}) \) factor more carefully.

In old books on relativity, one sees in discussions of the Riemann curvature tensor \( R_{ijkl} \) the following:

1) \( R_{ijjl} = R_{jikk} = R_{jikl} \)

2) \( R_{ijjl} = R_{klij} \)

3) \( R_{ijjl} + R_{jikl} + R_{kijl} = 0 \) (Voigt notation?)

Since \( R_{ijkl} \) is an element of \( V \otimes V \otimes V \otimes V \) (for \( V = \) tangent space)

Statement 1) is just saying that \( R \) is actually in \( S^2 V \otimes S^2 V \)

2) is “ ‘ ‘ ‘ \( R \) is actually in \( \frac{S^2(S^2 V)}{S^2(S^2 V)} \) —

This situation is of our interest.

Note the third line (Blaschke notation) is more interesting.

In the presence of other symmetries, it is easy to see that 3) is equivalent to:

\[
\sum \text{All Permutations} R_{ijkl} = 0
\]

But the left side is just the projection of \( V^4 \) to \( S^4 V \); so we have:

\( R \) is an element of \( \frac{S^2(S^2 V)}{S^2(S^2 V)} \) which is in the kernel of the natural projection \( \frac{S^2(S^2 V)}{S^4 V} \)

It is a classical result that this kernel is an irreducible GL-module and corresponds to the Young diagram \( \square \).
This is of interest to us, since if we look at our formulas for \( \sigma(T_{\lambda_1}) \) and \( \sigma(T_{\lambda_2}) \) and project them to \( \mathbb{S}^4 \mathbb{H} \) (i.e., treat them as polynomials), we see immediately that they go to \( C \) in other words:

\[
T = \lambda_0 \sigma + \mathbb{D} \oplus \mathbb{D}
\]

If our aim is now to show how that \( C \) is actually an equality:

\[
T_{\lambda_0} \lambda_0 \sigma = \mathbb{D} \oplus \mathbb{D}, \text{ and so also}
\]

\[
T_{\lambda_0} \mathbb{D} = \mathbb{D} \oplus \mathbb{D} - (\mathbb{D} \oplus \mathbb{D})^* = \mathbb{D} \oplus \mathbb{D}
\]

This is our main result quoted at the beginning:

Since the classical formula for \( \dim \mathbb{D}(V) \) when \( \dim V = n \) is

\[
\frac{n^2(n^2 - 1)}{12}, \text{ and in our case } V = H \text{ not } V = 2g
\]

we get

\[
\dim \mathbb{D}(V) = \frac{g^2(4g^2 - 1)}{3} + 1; \quad \text{as stated in the forward.}
\]

It remains to prove the theorem. To this end, we must look at how \( C \), which is \( GL(2g) \)-decomposable, decomposes.

The symplectic group \( \mathbb{S}^2 \).

We have noted that \( \mathbb{S}^2 \) acts affinely linearly on \( T \). This implies that \( \mathbb{S}^2 \) acts linearly on \( \mathbb{L}(\mathbb{S}^2) \) and \( \mathbb{S}^2 \mathbb{L}(\mathbb{S}^2) \) and it is easy to see that it preserves homogeneity.

Also, \( T = \mathbb{D} \oplus \mathbb{D} \) and we have seen that \( \mathbb{D} \cong [T] \).

Hence conjugation by Torelli implies no change on \( T \), i.e., \( T \) is odd on \( \mathbb{S}^2 \)-module and the action is preserved by the identification \( T \).

Thus, \( T = \lambda_0 \sigma \) is an \( \mathbb{S}^2 \)-submodule of \( \mathbb{D} \oplus \mathbb{D} \).
PROVE THE EQUATION FIRST. BECAUSE IT SUFFICES FOR THE CASE WHERE S = 0. 

PROOF: \[ \sum \text{Problems (cont.)} \]

THEOREM 3: \( \sum \text{Problems (cont.)} \)

WE WILL GREATLY AID US IN DETERMINING \( T \).

\[ \text{Our second classical result is that the GL irreducibles reduce completely over } G \text{ or } J \text{ by the process of contraction, which we now explain.} \]

Given a block from \( B \) on \( V \) and \( \text{conj} \) on \( V \), we define a map \( C_{ij} : V^m \to V^{m-2} \) by

\[ C_{ij}(x_1, x_2, \ldots, x_j, \ldots, x_m) = B(x_i, x_j) x_2, \ldots, x_m \]

Note that if \( B \) is symmetric, then \( C_{ij} = C_{ij}^t \).

Also, if \( C_{ij} \) is symmetric on \( (i, j) \), then \( C_{ij}(g) = C_{ij}(g^t) \).

Assume if \( \Theta = V \), not the opposite symmetry on indices \( C \) from \( B \).

Thus \( C_{ij}(g) = 0 \).

Note: if \( GG \) is GL, then \( B \) is not the contraction map and \( G \)-homogeneous. So \( \text{Ker} \) \( C_{ij} \) and \( \text{Conj} \) \( G \).

As an example, consider the subgroup \( \Theta \in H^4 \). For \( \Theta \in \Theta \).

We have symmetric \( (12), (34), (13)(24)_j \).

Our bilinear form \( U \) is the (symmetric) injection from \( J \).

Hence we get:

\[ C_{12} = C_{34} = 0 \]
\[ C_{13} = C_{14} = C_{24} = C_{23} \]

So there is only one contraction possible for \( \Theta \) on \( H \) with its natural action induced by \( H \).

The classical reduction theorem for \( GL \) irreducibles now is a space with \( \Theta \) symmetric bilinear form. This says that it "decomposes".

\[ \Theta \]

By means of contractions, in our case with only one contraction we get:

\[ W = \text{Ker } C_{13} : H^4 \to H^4 \]

is irreducible (this is the Weyl tensor). Therefore, any decompose further.

Our form is to decompose \( \Theta \) completely into irreducibles. We will find:

\[ \Theta = W + J + \Theta \]

where \( J \) is irreducible.

Thus \( \text{Ker} (J : \Lambda^2 \to \Theta) \) is unknown and irreducible.
We first show the reduction theorem works in a simple but useful case, namely $\Lambda^2 \subset \Lambda^2 H$. In this case there is only one contraction $\mathbf{C}_{12}$, and $\mathbf{C}_{12}(\Lambda^2 H) = \mathbf{C}_{12}(\frac{\Lambda x y - \Lambda y x}{2}) = \frac{\Lambda x y - y x}{2}$.

Thus, $\mathbf{C}_{12}$ is the same as our map $\mathbf{J}$, the intersection form on $\Lambda^2$.

The kernel of $\mathbf{C}_{12}$ is what we have been calling $J_0$, and $J_0$ is thus irreducible over $\mathfrak{sp}$. To see the splitting of $\Lambda^2$ into $J_0 \oplus \Theta$, note that the element $\Theta = \sum_{i=1}^{n} \Lambda^{i} \Lambda^{i}$ for $\Theta \in \Lambda^2$ is a symmetric basis of $H$, is an invariant of $\Lambda^2$ in $\mathfrak{sp}$, and that $\mathbf{J}(\Theta) = 1$ in $\mathfrak{g}$. Hence $\Lambda^2$ splits as $J_0 \oplus \Theta$, where $\Theta$ is the one-dimensional invariant subspace generated by the $\Theta$, or, a representation map $\Theta$ is trivial, i.e., $\Theta = 0$.

As a corollary, we have:

If $\alpha, \beta \in H$ then $\alpha \beta$ generates $\Lambda^2 H$ over $\mathfrak{sp}$.

Proof: $\mathbf{J}(\alpha \beta) = 1$ so $\alpha \beta \in J_0$. Also, for $g = 2$, $\alpha \beta \in g(\Theta)$.

Hence $\alpha \beta$ projects nontrivially into both summands $J_0$ and $\Theta$.

Note: This corollary can be proved directly and by simply moving $\alpha \beta$ around by $\mathfrak{sp}$ and setting a basis for $\Lambda^2$ over $\mathfrak{sp}$ in terms of $\beta = 0$.

Now we look at the reduction of $H$ using the contraction $\mathbf{C}_{3}: H \rightarrow \Lambda^2 H$.

We show that $\Lambda^2 \mathbf{C}_{3} = \Lambda^2 H$, which by the reduction theorem (proved earlier) must be $\Lambda^2 \mathbf{C}_{3} = \Lambda^2 H$.

$\mathbf{W} \oplus \Lambda^2 H$, where $W = \text{ker} \mathbf{C}_{3}$ (Note: $W$ will turn out to be called Weyl).

This gives us now the complete reduction of $H$ into irreducible $\mathfrak{sp}$.

$\mathfrak{sp} \oplus J_0 \oplus \Theta$.

Actually, we can show that $\mathbf{C}_{3}(\overline{S^2(H)} \subset \Lambda H)$, so this is certainly true of $\mathfrak{sp} \subset S^2(\overline{S^2(H)} < S^2(H)$. To see this,

Note that $S^2(\overline{S^2(H)})$ is generated by $\overline{S^2(H)} = \mathbb{C} \Lambda^2 H$. 

\[ \overline{S^2(H)} = \mathbb{C} \Lambda^2 H \]
\[(a \otimes b) \circ (c \otimes d) = \frac{1}{2} \left( (a \otimes c) \circ (b \otimes d) + (a \otimes d) \circ (c \otimes b) \right).\]

Applying \(C_{13}\) we get
\[2(a \otimes c)(b \otimes d) + (c \otimes a)(d \otimes b) = (a \otimes c) \otimes_{\text{N}} \otimes_{\text{N}} (b \otimes d) = (a \otimes c) \otimes (b \otimes d),\]

so
\[C_{13}(a \otimes b) \circ (c \otimes d) = (a \otimes c) \otimes (b \otimes d) \in \mathbb{N}^2.\]

To see that \(C_{13}(\otimes)\) is actually equal to \(\mathbb{N}^2\), we apply \(C_{13}\) to the element \((a \otimes b)(a \otimes b) - (a \otimes b)(a \otimes b)\) where \(a \otimes b = 1\) in \(\mathbb{N}^2\).

Note that this is just the projection of \(\sigma(T_g) \in \mathbb{R}^2\) to \(\mathbb{N}^2\), where \(T_g\) is some BSSC mapping a surface of genus \(g\) to \(\mathbb{N}^2\).

We get:
\[a \otimes b = a \otimes a \quad b \otimes b = b \otimes b \quad a \otimes b = \frac{a \otimes b + b \otimes a}{2},\]

so
\[C_{13}(a \otimes b, b \otimes c) = \frac{1}{2} \left( (a \otimes b)(b \otimes c) + (b \otimes c)(a \otimes b) \right)\]

and
\[(a \otimes b) \otimes (b \otimes c) = \frac{3}{4} (a \otimes b)(b \otimes c) + \frac{3}{4} (b \otimes c)(a \otimes b),\]

since \(C_{13}(\otimes)\) is an \(S^2\)-derivative of \(\mathbb{N}^2\) and contains \(\frac{3}{4} a \otimes b,\)

by the previous corollary we have \(C_{13}(\otimes) = \mathbb{N}^2\).

We proceed to the proof of the main theorem:

\[\tau = \text{im} \sigma = \mathbb{N} \oplus \mathbb{N}\]

i.e., \(\sigma : \mathbb{R} \rightarrow \mathbb{N} \oplus \mathbb{N}\) is surjective with \(\mathbb{Q}\) onto.

Since \(\mathbb{N} \oplus \mathbb{N}\) splits irreducibly into \(W \oplus \mathbb{N} \oplus \mathbb{N} \oplus \mathbb{N}\), it is enough

To show that:
1. \(L \otimes \otimes\) projects onto \(W \oplus \mathbb{N} \oplus \mathbb{N} \oplus \mathbb{N}\)
2. \(L \otimes \otimes\) projects onto \(\mathbb{N} \oplus \mathbb{N}\)

The above imply that \(L \otimes \otimes\) contains \(W\) and \(\mathbb{N} \oplus \mathbb{N}\), so coupled with \(b\)

we have \(L \otimes \otimes\) on \(\mathbb{N} \oplus \mathbb{N}\). (proof continues...)

We use already proved \(\frac{3}{4} (a \otimes b)\). To prove \(a\), we have already seen that \(\sigma(T_g) = \frac{3}{2} (a \otimes b)\).
So that $L = \mathbb{Z}/N$, project down to $\mathbb{F} = \mathbb{Z}/N$, only project onto the $\mathbb{A}^2$.

Now, if we can find an element of $\mathbb{F}$ which satisfies the above, then in $W$, we will have found $\alpha$. To do this, we construct a $\alpha$ in $\mathbb{F}$ which gives $0$ in $\mathbb{A}^2$, i.e., for which $\alpha_3 = 0$, but which then to $\mathbb{F}$.

Let $\bar{f}_i = \bar{g}_i$ under $\bar{Z}$ be given by $\mathbb{F}$'s resolving scheme. $\bar{f}_i$ will.

Note: This can be done in any surface of genus $\geq 2$.

Computing $\sigma(f_i)$ we get respectively:

\[
\sigma(f_1) = \left( \frac{g}{\rho a_1}, \frac{h}{b_1} \right) - \left( \frac{g}{\rho b_1}, \frac{h}{b_1} \right)^2 + 1
\]

\[
\sigma(f_2) = \left( \frac{(a_2 + b_2)(a_2 + b_2)}{b_1}, \frac{h}{b_1} \right) - \left( \frac{g}{\rho b_1}, \frac{h}{b_1} \right)^2 + 1
\]

Note above 2 with $f_1 \leftrightarrow f_2$

Now $\sigma(f_1 f_2 f_3 f_4^{-1})$ has no $\sigma_f$ where $f_i$ in $\mathbb{F}$.

And apply $\pi_3$ to it we get:

\[
\pi_3 \left( \sigma f_2 + \sigma f_3 - \sigma f_1 - \sigma f_4 \right) = \frac{1}{2} \sum \left( a_2 b_2 h_1 + (a_2 + b_2) h_2 - a h_1 - b h_2 \right)
\]

It remains to see that it is $W$.

Calculating $\sigma f_2 + \sigma f_3 - \sigma f_1 - \sigma f_4$ we get:

\[
\left( \frac{g}{\rho a_1} + \frac{2(a_2 b_2) + (a_2 + b_2)}{h_2} \right) + \left( \frac{g}{\rho b_1} - \frac{(a_2 b_2)}{h_1} \right) - 2 \left( \frac{(a_2 b_2)}{h_1} \right)^2
\]

\[
+ \left( \frac{g}{\rho a_2} + \frac{2(a_2 b_2) + (a_2 + b_2)}{h_2} \right) + \left( \frac{g}{\rho b_1} - \frac{(a_2 b_2)}{h_1} \right) - 2 \left( \frac{(a_2 b_2)}{h_1} \right)^2
\]

\[
- \left( \frac{g}{\rho a_1} \right)^2 + \left( \frac{g}{\rho b_1} \right)^2
\]

\[
- \left( \frac{g}{\rho a_2} \right)^2 + \left( \frac{g}{\rho b_1} \right)^2
\]

\[
= 2 \sum \left( a_2 b_2 h_1 + (a_2 + b_2) h_2 - (a_1 b_1) \right)
\]

This is zero in $\mathbb{A}$ to prove because an the $\mathbb{A}$ which shows that they must cancel.
This completes the proof of a). It remains to prove b), that the group of 2, 3 isomorphic.

As we have seen from the reasoning of 1 and 2, the projection of 2, 3 onto 2 is given by $G_{33} \rightarrow G_{22} \rightarrow G_{22}$. Now we determine the

projection to the first $G_{22}$, which is given by

$G_{33} \rightarrow G_{22} \rightarrow G_{22}$.

We have already seen that a group 3 map reducing a surface with boundary to $\frac{3}{2} a \wedge b$ in $\mathbb{R}^2$ is $a \wedge b$ to $\frac{3}{2} a \wedge b$ in $\mathbb{Q}$. To see the

image of a group 2 map, e.g.

$G_{33} (a, b, c, 0, 0) = (a, b, c) \rightarrow (a, b, c) + \frac{3}{2} (a, b, c, 0, 0)$

we must first evaluate $G_{33}$ on the left part (i.e., ignore the $2^n$).

on the right the bracket we get $\frac{3}{2} a \wedge b + \frac{3}{2} a \wedge b$.

Now we use

\[ G_{33} \left( (a, b, c, 0, 0) \right) = \left( \frac{\left( a, 0, b, c, 0, 0 \right)^2}{2}, \frac{\left( b, 0, a, c, 0, 0 \right)^2}{2} \right) \]

\[ = \frac{1}{4} \left( a, b, c, 0, 0, 0 \right) + \frac{1}{4} \left( 0, 0, 0, a, b, c \right) \]

\[ = \frac{1}{4} \left( a, b, c, 0, 0, 0 \right) + \frac{1}{4} \left( 0, 0, 0, a, b, c \right) \]

\[ = \frac{1}{4} \left( a, b, c, 0, 0, 0 \right) + \frac{1}{4} \left( 0, 0, 0, a, b, c \right) \]

\[ = \frac{1}{4} \left( a, b, c, 0, 0, 0 \right) + \frac{1}{4} \left( 0, 0, 0, a, b, c \right) \]

\[ = \frac{1}{4} \left( a, b, c, 0, 0, 0 \right) + \frac{1}{4} \left( 0, 0, 0, a, b, c \right) \]

So the total result is

\[ \frac{3}{2} (a, b, c, 0, 0) + 2 \left( \frac{a, b, c, 0, 0}{4} - \left( - \frac{a, b, c, 0, 0}{4} \right) \right) = 5 \]

\[ G_{33} \left( \text{Group 2} \right) = \frac{5}{2} (a, b, c, 0, 0) \]

Applying $J$ to this, we get $\sigma$ (Group 2) projects to $\frac{10}{2} = 5$ in the first $G_{22}$.

We thus have the projections of Group 2 into $\mathbb{Q}$ are

\[ (\frac{3}{2}, 1) \quad \text{and} \quad (\frac{3}{2}, 2) \]

respectively. Since $\sigma$ (Group 2) $= \frac{1}{2} \left( 1, 1 \right) = -2$.

We thus that the projection $\mathbb{Q}$ is all of $\mathbb{Q}$. This proves b) and hence the main theorem.
Remark: (Unpublished Work from 76) The projection of $\widehat{\mathbb{R}}$ to $\mathbb{R}$.

In particular, the composition map $\mathcal{J} \overset{\mathcal{J}}{\rightarrow} \mathcal{J} \overset{\mathcal{J}}{\rightarrow} \mathcal{J}$ has a description in terms of the action of $\mathcal{J}$ on $\pi_i / \left[\pi_i, [\pi_i, \pi_i] \right]$. In particular, (a) $\mathcal{J}$ acts trivially on $\pi_i / \left[\pi_i, [\pi_i, \pi_i] \right]$.

(b) Hence $f \in \mathcal{J}$, $x \in \pi_i$, $f(x) x^t \in \pi_i^{(3)}$.

In fact, $x \in \pi_i^{(3)} / \pi_i^{(0)}$, $f(x) x^t \in \pi_i^{(3)} / \pi_i^{(0)}$, which is abelian, in fact a finite abelian group.

c) The map $x \mapsto f(x) x^t$ is a homomorphism $H \rightarrow \pi_i^{(3)} / \pi_i^{(0)}$.

d) The map $\mathcal{J} \rightarrow \text{Hom}(H, \pi_i^{(3)} / \pi_i^{(0)})$ is not surjective.

A homomorphism $\mathcal{J} \rightarrow \text{Hom}(H, \pi_i^{(3)} / \pi_i^{(0)}) \cong H^* \otimes \pi_i^{(3)} / \pi_i^{(0)}$ can be identified with $\mathcal{J}$ and $\pi_i^{(3)} / \pi_i^{(0)}$ identified with $\mathbb{R}$.

e) The image of $\mathcal{J}$ is in $\mathbb{R}$ and $\mathcal{J}$ (see 1.2) has some properties as far as we see.

Thus identifying $\mathcal{J}$ and $\pi_i^{(3)} / \pi_i^{(0)}$...