TOPIC: THE SPACE OF CASSON HOMOMORPHISMS FOR A SURFACE $K_{g,1}$

THE IDEA: $\mathcal{M}$ = MAPPING CLASS GROUP OF $K_{g,1}$, $H = H_1(K_{g,1}, Z)$; INTERSECTION $J: H\otimes H$

$\mathcal{I}$ = TORELLI $(K_{g,1})$ $\mathcal{J} \subset \mathcal{I}$ = GROUP GEN BY TWISTS ON BOUNDING CURVES IN K

THEN $\exists$ A FAMILY OF HOMS $\lambda_L : \mathcal{J} \to Z$ DEFINED BY CASSON'S INVARIANT.

THE PARAMETER L RANGES OVER THE LINKING FORMS ON K, THAT IS

BILINEAR $L: H\otimes H \to Z$ SUCH $L(x,y) - L(y,x) = x\cdot y$, I.E.

$L - L^T = J$

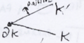
WE WILL FIND THE STRUCTURE AND RANK OF THE ABELIAN GROUP Λ OF HOMS $\mathcal{J} \to Z$ GENERATED BY THE λ_L: ALTHOUGH THE λ_L ARE ∞ IN #, Λ IS FREE ABELIAN OF FINITE RANK $1 + \frac{g^2(4g^2-1)}{3}$.

ALSO, Λ IS ISOMORPHIC TO $\oplus$ A CERTAIN GL-IRREDUCIBLE TENSOR CONSTRUCTION ON H WHICH IS EQUIVALENT TO THE RIEMANN CURVATURE TENSOR.

I. THE CASSON HOM λ_K FOR $K=K_{g,1}$ A SURFACE S^3

① FOR $K=K_{g,1}\subset X^3$ HOMOLOGY SPHERE AND f A DIFFEO OF K, FORM A 3-MANIFOLD $Y^3(K,f)$ BY:

A. SLICE S^3 ALONG K:



B. REGLUE K TO K' BY IDENTIFYING S^3

x IN K WITH $f(x)$ IN K'.

NOTE: IF $f=T_\gamma$, WHERE γ IS A BSCC IN K, THEN $Y(K,T_\gamma)$ IS THE HOMOLOGY SPHERE GOT BY DOING $1/1$ (-~~2~~) SURGERY ON $\gamma\subset X^3$.

ALSO, BY CASSON, $Y(K,T_\gamma)=\lambda'(\gamma\subset S^3)$

FOR $f\in\mathcal{I}(K)$, DEFINE $\lambda_K(f)$ TO BE $\lambda(Y(K,f))$

② IF $f=\prod_{i=1}^{n}T_{\gamma_i}$ (γ_i BSCC'S IN K) A THEN $\lambda_K(f)=\sum_{i=1}^{n}\lambda'(\gamma_i\subset S^3)$

PROOF: I'LL DO $n=2$ & THE REST WILL BE OBVIOUS:

GLUING BY $T_{\gamma_1}T_{\gamma_2}$ ~~[illegible]~~ IS THE SAME AS DOING A SURGERY ON $\gamma_2\subset K$ AND ANOTHER ON ~~[illegible]~~ $\gamma_1^+\subset K^+$ GOT BY PUSHING K UP IN THE POSITIVE DIRECTION:

THE GLUING DONE IS BY T_{γ_2} AS WE PASS UPWARD THROUGH K AND $T_{\gamma_1^+}$ AS WE PASS THROUGH K^+

DOING THE γ_2 SURGERY, WE GET

$$\lambda_K(T_{\gamma_2})=\lambda(Y(K,T_{\gamma_2}))\overset{\text{BY CASSON}}{=}\lambda'(\gamma_2\subset S^3)$$

DOING THE SECOND SURGERY, WE GET

$$\lambda_K(T_{\gamma_1}T_{\gamma_2})=\lambda(Y(K,T_{\gamma_1}T_{\gamma_2}))\overset{\text{BY CASSON}}{=}\lambda'(\gamma_1^+\text{ IN }Y(K,T_{\gamma_2}))+\lambda(Y(K,T_{\gamma_2}))$$

$$=\lambda'(\gamma_1^+\text{ IN }Y(K,T_{\gamma_2}))+\lambda'(\gamma_2\text{ IN }S^3)$$

$$\overset{\text{BY CASSON}}{=}\lambda'(\gamma_1^+\text{ IN }S^3)+\lambda''(\gamma_1^+,\gamma_2)+\lambda'(\gamma_2\text{ IN }S^3)$$

BUT $\gamma_1^+ \cup \gamma_2$ IS A BOUNDARY LINK, SINCE γ_1^+, γ_2 BOUND DISJOINT SURFS IN K^+, K, SO $\lambda'' = 0$ AND GET

$$\lambda_K(T_{\gamma_1} T_{\gamma_2}) = \lambda'(\gamma_1^+) + \lambda(\gamma_2) = \lambda'(\gamma_1) + \lambda'(\gamma_2) =$$

$$= \lambda_K(T_{\gamma_1}) + \lambda_K(T_{\gamma_2}) \quad \text{QED}$$

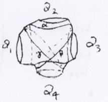
THIS SHOWS ALSO THAT $\lambda_K(fg) = \lambda_K(f) + \lambda_K(g)$ FOR $f, g \in \mathcal{J}$, SO λ_K IS A HOM $\mathcal{J} \to \mathbb{Z}$.

③ TO SEE HOW λ_K BEHAVES ON $\mathcal{J}$, WE WILL COMPUTE IT ON GENERATORS FOR $\mathcal{J}(K)$. $\mathcal{J}$ IS GENERATED BY BSCC CURVES, BUT WE CAN DO BETTER: $\mathcal{J}$ IS GEN BY BSCC CURVES OF GENUS 1 AND 2, I.E. CURVES WHICH BOUND A SUBSURFACE OF K OF GENUS 1 OR 2.

<u>PROOF</u>: WE START WITH THIS RELATION ON A DISK WITH 3 HOLES (MOD ∂):

THEN $T_\gamma T_\beta T_\alpha = T_{\partial_1} T_{\partial_2} T_{\partial_3} T_{\partial_4} \pmod{\partial}$

THE ABOVE PICTURE DEFORMS TO



NOW USE PICTURE

TO GET T_{h+2} A PROD OF T_{h+1}'s T_h's, T_2's, T_1's.

QED.

SO WE NEED ONLY CALCULATE λ_K ON GENUS 1 & 2 GENERATORS.

④ TO CALCULATE $\lambda_K(T_\gamma)$ WE NEED THE ALEXANDER POLY OF γ, WHICH WE CAN COMPUTE FROM THE LINKING FORM OF A SEIF SURF FOR γ. SINCE γ BOUNDS <u>IN K</u>, WE HAVE A (UNIQUE) SUBSURFACE S OF K AS A SEIF SURF FOR γ, AND CAN USE THE LINKING FORM L OF $K \subset S^3$, RESTRICTED TO S. WE DO THIS NOW FOR GENUS 1 & 2 CURVES γ.

~~[illegible]~~:

⑤ COMPUTATION OF ALEXANDER 2ND DERIVS:

$\boxed{\text{GENUS } 1}$: $L = \begin{pmatrix} L(a,a) & L(a,b) \\ L(b,a) & L(b,b) \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$

ALEXANDER $= at^2 + bt + a = \det(tL - L^T)$ WHERE $a = \det L$

SYM ALEX $= at + b + at^{-1}$ $\therefore \frac{1}{2}D^2(\text{ALEXANDER}) = \frac{1}{2}(2a) = a = \det L = \begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix}$

$\boxed{\text{GENUS } 2}$: $L = (L_{ij})$ ALEX $= \det(tL - L^T) = at^4 + bt^3 + ct^2 + bt + a$

SYM ALEX $= at^2 + bt + c + bt^{-1} + at^{-2}$ $\frac{1}{2}D^2(\text{SYM}) = \frac{1}{2}(2a + 2b + 6a) = 4a + b$

$a = \det L$ $b = \sum_i \det(\text{REPLACE COL}_i \text{ OF } L \text{ BY } -L^T)$

So $4a + b = \sum \det(\text{SUBTRACT COL}_i \text{ OF } L^t \text{ FROM COL}_i \text{ OF } L)$

SINCE $L - L^T = J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ EACH OF THE ABOVE IS EVALUATED EASILY AS A 3x3 DET:

$$4a+b = \begin{vmatrix} L_{12} & L_{13} & L_{14} \\ L_{32} & L_{33} & L_{34} \\ L_{42} & L_{43} & L_{44} \end{vmatrix} - \begin{vmatrix} L_{21} & L_{23} & L_{24} \\ L_{31} & L_{33} & L_{34} \\ L_{41} & L_{43} & L_{44} \end{vmatrix} + \left(\begin{array}{l} \text{DO } 1 \leftrightarrow 3 \\ 2 \leftrightarrow 4 \\ \text{ON INDICES} \\ \text{ALSO SWITCH CORRESP} \\ \text{ROWS \& COLS:} \\ \text{NO CHANGE IN ANS} \end{array}\right)$$

$\downarrow$ TRANSPOSE

$$\parallel$$

$$\begin{vmatrix} 1 + L_{21} & L_{13} & L_{14} \\ L_{23} & L_{33} & L_{34} \\ L_{24} & L_{43} & L_{44} \end{vmatrix} - \begin{vmatrix} L_{21} & L_{13} & L_{14} \\ L_{23} & L_{33} & L_{43} \\ L_{24} & L_{34} & L_{44} \end{vmatrix}$$

$$\parallel$$

$$\begin{vmatrix} L_{33} & L_{34} \\ L_{43} & L_{44} \end{vmatrix} + \underbrace{\begin{vmatrix} L_{21} & L_{13} & L_{14} \\ L_{23} & L_{33} & L_{34} \\ L_{24} & L_{43} & L_{44} \end{vmatrix} - \begin{vmatrix} L_{21} & L_{13} & L_{14} \\ L_{23} & L_{33} & L_{43} \\ L_{24} & L_{34} & L_{44} \end{vmatrix}}$$

EXPAND BY TOP ROW: L_{21} TERMS CANCEL

$$-L_{13}\left(\begin{vmatrix} L_{23} & L_{34} \\ L_{24} & L_{44} \end{vmatrix} - \begin{vmatrix} L_{23} & L_{43} \\ L_{24} & L_{44} \end{vmatrix}\right) = -L_{13}\left(\begin{vmatrix} L_{23} & 1 \\ L_{24} & L_{44} \end{vmatrix}\right) = L_{13}L_{24}$$

$$L_{14}\left(\begin{vmatrix} L_{23} & L_{33} \\ L_{24} & L_{43} \end{vmatrix} - \begin{vmatrix} L_{23} & L_{33} \\ L_{24} & L_{34} \end{vmatrix}\right) = L_{14}\begin{vmatrix} L_{23} & L_{33} \\ L_{24} & -1 \end{vmatrix} = -L_{14}L_{23}$$

SO GET: $\begin{vmatrix} L_{13} & L_{23} \\ L_{14} & L_{24} \end{vmatrix}$: TOTAL $\begin{vmatrix} L_{33} & L_{34} \\ L_{43} & L_{44} \end{vmatrix} + \begin{vmatrix} L_{13} & L_{23} \\ L_{14} & L_{24} \end{vmatrix} + (\text{DO } (13)(24))$: $\begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix} + \begin{vmatrix} L_{33} & \\ L_{43} & L \end{vmatrix} + 2\begin{vmatrix} L_{13} & L_{23} \end{vmatrix}$

⑥ NOW LET K BE AN ABSTRACT SURFACE OF TYPE $(g,1)$ (NOT IMBEDDED IN S^3) FOR ANY IMBEDDING φ OF K INTO S^3, WE GET A CASSON HOM $\lambda_\varphi : \mathcal{J}(K) \to Z$ WHICH IS INDUCED FROM $\varphi(K) \subset S^3$: THAT IS, FOR $f \in \mathcal{J}(K)$ $\varphi f \varphi^{-1}$ IS IN $\mathcal{J}(\varphi(K))$ SO WE CAN DEFINE $\lambda_\varphi(f) = \lambda_{\varphi(K)}(\varphi f \varphi^{-1})$. THIS HAS THE FOLLOWING CONSEQUENCES:

A) IF $h \in \mathcal{M}(K)$ THEN φh IS ALSO AN IMBEDDING OF $K \hookrightarrow S^3$, WITH SOME IMAGE AND $\lambda_{\varphi h}(f) = \lambda_{\varphi(K)}(\varphi h f h^{-1} \varphi^{-1}) = \lambda_\varphi(h f h^{-1})$

[illegible] I.E. $\boxed{\lambda_\varphi(hfh^{-1}) = \lambda_{\varphi h}(f)}$

B) THE VALUE OF $\lambda_\varphi(T_\gamma)$ IS CALCULATED BY ~~THE~~ USING THE LINKING
$\gamma \subset K$
FORM ON $\varphi(K)$, RESTRICTED TO THE SURFACE $\varphi(S)$ BOUNDED BY $\varphi(\gamma)$.
PULLING THIS FORM BACK VIA φ TO K GIVES US AN INDUCED FORM L_φ ON K FROM WHICH WE CAN ALSO COMPUTE THE SAME LINKING INFORMATION:
I.E. $\lambda_\varphi(T_\gamma)$ = THE VALUE GOT BY APPLYING THE L-FORMULAS TO S USING L_φ. NOTE: L_φ IS A LINKING FORM ON K: $L_\varphi - L_\varphi^T = J$

~~$C)\ \lambda_\varphi(hfh^{-1}) = \lambda_{\varphi h}(f)$~~

C) BY THE ABOVE, $\lambda_\varphi(f)$ DEPENDS ONLY ON L_φ, NOT ON φ ITSELF. FOR THIS REASON, WE CAN WRITE $\lambda_L(f)$ INSTEAD OF $\lambda\varphi(f)$.

NOTE: GIVEN ANY LINKING FORM ON K, THERE IS AN IMBEDDING $\varphi : K \hookrightarrow S^3$ INDUCING THIS FORM; HENCE FOR EVERY LINKING FORM L THERE IS A UNIQUE CASSON HOMOMORPHISM
$\lambda_L : \mathcal{J}(K) \to Z$.

D) $\underset{\substack{h \in \mathcal{M} \\ f \in \mathcal{J}}}{\lambda_L(hfh^{-1})} = \underset{\varphi \text{ INDUCING } L}{\lambda_\varphi(hfh^{-1})} = \lambda_{\varphi h}(f) = \lambda_{h^*L}(f)$

SINCE φh INDUCES h^*L
$[h^*L(x,y) \equiv L(hx,hy)]$

I.E. $\boxed{\lambda_L(hfh^{-1}) = \lambda_{h^*L}(f)}$

NOTE ALSO THAT THE ACTION OF h ON $\mathcal{J}$ FACTORS THROUGH Sp — THAT IS, $\mathcal{L}$ IS AN Sp-AFFINE SPACE: Sp ACTS ~~AFFINE~~ LINEAR ON $\mathcal{L}$.

SO ALSO; Sp ACTS ON THE CASSON HOMS

E) IF $h \in \partial$ THEN $h^*L = L$, SO GET:

$$\boxed{\lambda_L(hfh^{-1}) = \lambda_L(f) \quad \text{FOR } h \in \partial\text{, ALL } L}$$

IN PARTICULAR: $\boxed{\lambda_L[\partial, \mathcal{J}] = 0 \quad \text{FOR ALL } L}$

SO ALSO: SUBGP Λ OF $\mathrm{Hom}(\mathcal{J}/[\partial,\mathcal{J}], Z)$ GIVEN BY CASSON HOMS IS ACTED ON BY Sp, SINCE ACTION IS BY CONJUGATION + ∂ CONTRIBUTES TRIVIALLY.

① WE HAVE BEEN LOOKING AT $\lambda_L(f)$ AS A HOM $\mathcal{J} \to Z$, I.E. AS A FUNCTION OF f. TO GET SOME GLOBAL INFO ABOUT THE SPACE $\Lambda \subset \mathrm{Hom}(\mathcal{J},$ OF CASSON HOMS, WE NOW TAKE AN ADJOINT POINT OF VIEW: WE WILL FIX γ TO BE A CURVE OF GENUS 1 OR 2 AND THINK OF $\lambda_L(T_\gamma)$ AS A FUNCTION OF THE LINKING FORM L.

TO BEGIN: LET $\mathcal{L}$ = THE SPACE OF LINKING FORMS L ON K!

$$L - L^T = J$$

THE FIRST THING TO NOTICE ABOUT $\mathcal{L}$ IS THAT IT'S NOT A LINEAR SUBSPACE OF ALL BILINEAR FORMS $H^* \otimes H^*$ ON H; BUT IT IS INFACT AN AFFINE LINEAR SUBSPACE OF $H^* \otimes H^*$ I.E. IT DOES NOT PASS THROUGH THE ORIGIN OF $H^* \otimes H^*$

IN FACT WE HAVE:

A) IF L_1, L_2 ARE TWO LINKING FORMS, THEN $L_1 - L_2 = S$ IS SYMMETRIC: $S^T = S$, OR $S(x,y) = S(y,x)$

B) CONVERSELY IF S IS SYMMETRIC, AND L IS A LINKING FORM, THEN $L + S$ IS A LINKING FORM.

C) HENCE $\mathcal{L}$ IS JUST A TRANSLATE OF THE <u>SYMMETRIC</u> BILINEAR FORMS $S^2(H^*) \subset H^* \otimes H^*$

~~[illegible]~~

⑧ IF $f \in \mathcal{J}$, WE NOW WRITE $\sigma_f : \mathcal{L} \to Z$, DEFINED BY

$$\sigma_f(L) = \lambda_L(f) \quad : \sigma \text{ IS THE "ADJOINT" OF } \lambda .$$

~~LOOKING AT THE L-FORMULAS FOR~~

NOTE THAT $\sigma_{f_1 f_2}(L) = \lambda_L(f_1 f_2) = \lambda_L(f_1) + \lambda_L(f_2) = \sigma_{f_1}(L) + \sigma_{f_2}(L)$

AND HENCE:

$$\boxed{\sigma \text{ IS A HOMOMORPHISM FROM } \mathcal{J} \text{ TO THE VECTOR SPACE OF FUNCTIONS } \mathcal{L} \to Z}$$

LOOKING AT THE L-FORMULAS FOR GENUS 1 & 2 CURVES γ WE SEE THAT $\sigma(T_\gamma)$ IS ACTUALLY AN (AFFINE) <u>QUADRATIC</u> FUNCTION ON $\mathcal{L}$ (ALL THE ENTRIES ARE QUADRATIC IN THE COORDS L_{ij})

HENCE σ ACTUALLY MAPS <u>ALL</u> OF $\mathcal{J}$ INTO <u>QUAD</u> FUNCTIONS OF $\mathcal{L} \to Z$:

$$\sigma(\mathcal{J}) \subset \overset{\text{AFFINE}}{\text{QUADRATIC}} \text{ FUNCTIONS } \mathcal{L} \text{ TO } Z$$

I.E. $\boxed{\sigma : \mathcal{J} \to \mathrm{QUAD}(\mathcal{L}, Z)}$ NOTE THAT $\mathrm{QUAD}(\mathcal{L}, Z)$ IS A FREE ABELIAN GROUP OF FINITE RANK

⑨ WE NOW DEFINE C TO BE THE COMMON KERNEL OF ALL THE CASSON HOMS λ_L:

$C = \bigcap_{L \in \mathcal{L}} \mathrm{Ker}\, \lambda_L$, AND WE PUT $T = \dfrac{\mathcal{J}}{C}$, WHICH MAKES SENSE SINCE C IS CERTAINLY NORMAL IN $\mathcal{J}$ — IN FACT C CONTAINS $\mathcal{J}'$ SINCE EACH $\mathrm{Ker}\,\lambda_L$ DOES, SO T IS ABELIAN, AND EVERY λ_L FACTORS THROUGH T, AND HENCE ALSO EVERY HOM IN $\Lambda \subset \mathrm{Hom}(\mathcal{J}, Z)$. WE THUS HAVE

$$\Lambda \subset \mathrm{Hom}(T, Z) = T^* \quad ✓$$

THEN WE HAVE THE FOLLOWING:

A) σ_f IS THE ZERO FUNCTION $\mathcal{L}$ IFF $\sigma_f(L) = 0$ ALL L,
I.E. IFF $\lambda_L(f) = 0$ ALL L, I.E. IFF $f \in \ker \lambda_L$ ALL L
I.E. IFF $f \in \mathcal{C}$. THUS $\sigma: \mathcal{J} \to \text{QUAD}(\mathcal{L}, Z)$
ALSO FACTORS THROUGH T, AND $\sigma: T \to \text{QUAD}(\mathcal{L}, Z)$ IS $\underline{\underline{1-1}}$

B) THIS IMPLIES THAT T IS FINITE RANK FREE ABELIAN, SINCE $\text{QUAD}(\mathcal{L}, Z)$ IS SO. HENCE $\Lambda \subset T^*$ IS ALSO FINITE RANK FREE ABELIAN.

C) SUPPOSE RANK $\Lambda <$ RANK T^*: THEN THERE WOULD EXIST A $t \in T$, $t \neq 0$, SUCH THAT $\lambda(t) = 0$ FOR ALL $\lambda \in \Lambda$; BUT WE HAVE ALREADY SEEN THAT $\lambda(t) = 0$ $\forall \lambda \Rightarrow t = 0$ IN T.

HENCE:

$$\boxed{\begin{array}{l}\text{RANK } \Lambda = \text{RANK } T^* = \text{RANK } T \\ \text{AND } \Lambda \text{ IS FINITE INDEX IN } T^*\end{array}}$$

⑩ SINCE Λ IS VIRTUALLY DUAL TO T, WE WILL INSTEAD INVESTIGATE THE STRUCTURE OF T, IDENTIFIED VIA σ WITH A SUBGROUP OF THE Z-VALUED QUADRATIC FUNCTIONS ON THE AFFINE SPACE $\mathcal{L}$.

ALSO, WE NOW ABANDON OUR RESTRICTION TO Z-VALUES AND TENSOR EVERYTHING WITH $\mathbb{Q}$: THIS WILL SMOOTH OUT CERTAIN MESSY PROBLEMS WITH SPLITTING, ETC. THUS $T \otimes \mathbb{Q}$ IS NOW CONSIDERED AS A SUBSPACE OF $\text{QUAD}(\mathcal{L}, \mathbb{Q})$

⑪ TO BEGIN, WE NEED TO UNDERSTAND THE AFFINE LINEAR FUNCTIONS $\text{LIN}(\mathcal{L}, \mathbb{Q})$, WHICH WE WRITE AS $\mathcal{L}^*$ FOR SHORT. NOTE THAT THE VECTOR SPACE OF AFFINE LINEAR FUNCTIONS ON AN AFFINE SPACE OF DIMENSION n HAS DIMENSION $\underline{n+1}$, AND INCLUDES THE CONSTANT FUNCTIONS; THERE GENERALLY IS NO NATURAL WAY TO DEFINE THE "HOMOGENEOUS" LINEAR FUNCTIONS IN THIS SETTING. (OVER $\mathbb{Q}$, IN THE PRESENT SETTING, THERE IS)

RECALL THAT $\mathcal{L}$ IS AN AFFINE SUBSPACE OF BILINEAR FORMS $H^* \otimes H^*$, AND HENCE $H \otimes H$ GIVES US IN THE OBVIOUS WAY LINEAR FUNCTIONS ON $\mathcal{L}$.

AFFINE LINEAR FUNCTIONS: Let $\mathcal{S}$ = symmetric forms. The $\text{LIN}(\mathcal{L}, \mathbb{Q}) = \{ f: \mathcal{L} \to \mathbb{Q} \mid f(L) = g(L - L_0) + b$ for some $b \in \mathbb{Q}$, $L_0 \in \mathcal{L}$, $g \in \text{Hom}(\mathcal{S}, \mathbb{Q}) \}$.
Notice that if $L_1 = L_0 + S$, then $f(L) = g(L - L_0) + b = g(L - L_1 + S) + b = g(L - L_1) + g(S) + b$. Thus
$\text{LIN}(\mathcal{L}, \mathbb{Q}) = \{ f \mid f(L) = g(L - L_0) + b, b \in \mathbb{Q} \}$ is $\text{Dim}(\mathcal{S}, \mathbb{Q}) + 1 = \dim \mathcal{L} + 1$

$x\otimes y \in \mathrm{Lin}(\mathcal{L}, Q)$: let $b = L_0(x,y)$ and $g \in \mathrm{Hom}(\mathcal{S}, Q)$ be $g(S) = S(x,y)$. Then g is lin, and $x\otimes y(L) = L(x,y) = L(x,y) - L_0(x,y) + b = g(L - L_0) + b$.

$$(x\otimes y)(L) = L(x,y)$$

LET $\Lambda^2 H$ BE DEFINED AS THE SUBSPACE OF $H\otimes H$ GENERATED BY

$$x\wedge y \underset{\text{DEF.}}{=} \frac{x\otimes y - y\otimes x}{2}$$

, AND $\int^2 H$ DEFINED ~~AS~~ GEN BY

THE "MONOMIALS" $x\cdot y \underset{\text{DEF}}{=} \frac{x\otimes y + y\otimes x}{2}$, SO THAT $H\otimes H$ SPLITS

AS $\Lambda^2 H \oplus \int^2 H$, AND THIS SPLITTING IS NATURAL OVER THE ACTION OF $G\ell$ ON $H\otimes H$.

ALSO WE HAVE THE INTERSECTION MAP $J: H\otimes H \to Q$ GIVEN BY $x\otimes y \to x\cdot y$;

THIS ALSO SENDS $x\wedge y$ TO $x\cdot y$. and $x\circ y \mapsto 0$ since $\Lambda^2 H \xrightarrow{J} Q$ is onto.

LET $J_0 = (\ker J)\cap \Lambda^2 H$, SO THAT $\Lambda^2 H / J_0 \simeq Q$ and if $\theta = J_0$

[illegible] : if $\theta \in \Lambda^2 H$, $\theta(L)$ is constant on $L \in \mathcal{L}$, and θ [illegible] J_0

LEMMA: IF $\theta \in J_0$ THEN $\theta(L) = 0$ FOR ALL $L \in \mathcal{L}$, I.E.

$\theta = 0$ IN $\mathcal{L}^*$; ALSO, IF $\theta \in \Lambda^2 H$, THEN θ IS A CONSTANT FUNCTION ON $\mathcal{L}$.

Pf: $\theta = \sum x_i \wedge y_i$ IS IN J_0 IFF $\sum x_i \cdot y_i = 0$

HENCE $\theta(L) \underset{\text{DEF}}{=} \sum L(x_i \wedge y_i) = \frac{1}{2}\sum\big(L(x_i, y_i) - L(y_i, x_i)\big)$

$= \frac{1}{2}\sum(x_i \cdot y_i)$; $J(\theta)$ THIS IS CONSTANT ON $\mathcal{L}$, AND IS ZERO IFF θ IS IN J_0; QED

COROLLARY: $\frac{H\otimes H}{J_0}$ ACT AS LINEAR FUNCTIONS ON $\mathcal{L}$.

NOW $\frac{H\otimes H}{J_0} = \int^2 H \oplus \frac{\Lambda^2 H}{J_0} \overset{\text{AS FUNCTIONS}}{\cong} \int^2 H \oplus Q^{\wedge}$, SINCE: — THE SPLITTING IS $S\!p$ INVARIANT!

a) $\dim \mathcal{L}^2 = \dim \int^2 H$, so $\dim \int^2 H \oplus Q = \dim \mathcal{L} + 1$

b) ALL LINEAR FUNCTIONS ON $\mathcal{L}$ ARE REPRESENTED IN $\frac{H\otimes H}{J_0} = \int^2 H\oplus Q$

WE HAVE $\frac{H\otimes H}{J_0}$ IS THE SPACE $\mathrm{Lin}(\mathcal{L}, Q) = \mathcal{L}^*$

~~NOTE THAT, THOUGHT OF~~ ~~NOTE THAT, THE~~ [illegible] OF FUNCTIONS, THE ELEMENTS OF $\int^2 H \oplus Q$ WHICH ARE IN Q ("IN" Λ^2) ARE JUST THE CONSTANT FU.ns [illegible]

⑫ IF $x, y \in H$, WE LET $[x\,y]$ REPRESENT THE IMAGE OF $x\otimes y$

IN $\frac{H\otimes H}{J_0}$; $[x\,y]$ IS A LINEAR FUNCTION ON $\mathcal{L}$, AND $[x,y](L) = L(x,y)$.

SINCE $\frac{H\otimes H}{J_0} = \int^2 H \oplus Q$ WE CAN WRITE $[x\,y]$ AS:

$$[xy] = \mathrm{lin}\,(x\otimes y = x\circ y + x\wedge y) = \underset{\underbrace{\text{IN } S^2H}_{\text{"HOMOGENEOUS LINEAR" FUNCTIONS}}}{x\circ y} + \underset{\underbrace{\text{IN } \mathbb{Q}}_{\text{CONSTANT FUNCTIONS}}}{x\cdot y}$$

⑬ WE ARE INTERESTED IN THE QUADRATIC FUNCTION SPACE ON $\mathcal{L}$, I.E.

$$\mathrm{Quad}(\mathcal{L},\mathbb{Q}) = S^2(\mathcal{L}^*) = S^2(S^2H\oplus\mathbb{Q}) \overset{\text{SUB-SP!}}{=} \underbrace{S^2(S^2H)}_{\text{"HOMOG. QUAD"}} \oplus \underbrace{S^2H}_{\text{"HOMOG LIN"}} \oplus \underbrace{\mathbb{Q}}_{\text{CONSTANTS}}$$

AND IN PARTICULAR, IN THE IMAGE T OF THE MAP $\sigma : \mathcal{T} \longrightarrow S^2(S^2H)\oplus S^2H\oplus\mathbb{Q}$, WHICH WE HAVE IDENTIFIED WITH $\dfrac{\mathcal{T}}{C}$. TO DETERMINE THIS IMAGE MORE PRECISELY, WE LOOK AT THE IMAGES OF THE GENUS 1 & 2 GENERATORS.

FOR A GENUS 1 GENERATOR γ_1 BOUNDING A SURFACE WITH HOMOLOGY BASIS a, b, WE HAVE

$$\lambda_2(T_{\gamma_1}) = \begin{vmatrix} L(a,a) & L(a,b) \\ L(b,a) & L(b,b) \end{vmatrix} = L(a,a)\,L(b,b) - L(a,b)\,L(b,a)$$

$$= \big([aa][bb] - [ab][ba]\big)(L)$$: THAT IS,

THE FUNCTION $\sigma(T_{\gamma_1})$ ON $\mathcal{L}$ IS GIVEN BY THE QUADRATIC FUNCTION

$$\underbrace{[aa][bb] - [ab][ba]}_{\text{LIN FUNCTIONS}}$$

BUT, AS A FUNCTION, WE HAVE $[xy] = x\circ y + x\cdot y$

SO: $[aa] = a\circ a \qquad [bb] = b\circ b$

$[ab] = a\circ b + 1 \qquad [ba] = b\circ a - 1 = a\circ b - 1$

AND $$\boxed{\sigma(T_{\gamma_1}) = (a\circ a)(b\circ b) - (a\circ b)^2 + 1}$$

LIKEWISE, FOR A GENUS 2 GENERATOR T_{γ_2}, WE GET

$$\sigma(T_{\gamma_2}) = \{(a_1\circ a_1)(b_1\circ b_1) - (a_1\circ b_1)^2\} + \{\text{DITTO } a_2\, b_2\} + 2\{(a_1\circ a_2)(b_1\circ b_2) - (a_1\circ b_2)(b_1\circ a_2)\} +$$

⑭ THE ABOVE FORMULAS SHOW THAT THERE IS "NO LINEAR PART" TO $\sigma(T_{\gamma_1}), \sigma(T_{\gamma_2})$ i.e. THEY LAND IN $S^2(S^2\hat{H}) \oplus Q$

AND HENCE $\boxed{T = \operatorname{Im}\sigma \subset S^2(S^2\hat{H}) \oplus Q}$ ← sp summand?

WE NOW MUST INVESTIGATE THE $S^2(S^2H)$ FACTOR MORE CAREFULLY

IN OLD BOOKS ON RELATIVITY, ONE SEES ~~the~~ IN DISCUSSIONS OF THE RIEMANN CURVATURE TENSOR R_{ijhl} THE FOLLOWING:

1) $R_{ijhl} = R_{jihl} = \cancel{\ } R_{ijlh}$

2) $R_{ijhl} = R_{hlij}$

3) $R_{ijhl} + R_{ihlj} + R_{iljh} = 0$ (whose identity?)

SINCE R_{ijhl} IS AN ELEMENT OF $V \otimes V \otimes V \otimes V$ (FOR $V =$ SOME TANGENT SPACE) STATEMENT 1) IS JUST SAYING THAT R IS REALLY IN $S^2V \otimes S^2V$

AND 2) IS " " " R IS ACTUALLY IN $S^2(S^2V)$ — THE SITUATION OF OUR INTEREST.

NOW THE THIRD LINE ($\overset{\text{NOT TRUE?}}{\boxed{\text{BIANCHI IDENT}}}$) IS MORE INTERESTING. IN THE PRESENCE OF ~~the~~ THE OTHER SYMMETRIES, IT IS EASY TO SEE THAT 3) IS EQUIVALENT TO:

$$\sum_{\substack{\text{ALL PERMUTATIONS}\\ \text{OF INDICES}}} R_{ijhl} = 0$$

BUT THE LEFT SIDE IS JUST THE PROJECTION OF V^4 TO S^4V; SO, IN MODERN TERMS WE HAVE: R IS AN ELEMENT OF $S^2(S^2V)$ WHICH IS IN THE KERNEL OF THE NATURAL PROJECTION $S^2(S^2V) \to S^4V$

IT IS A CLASSICAL RESULT THAT THIS KERNEL IS AN IRREDUCIBLE Gl-MODULE AND CORRESPONDS TO THE YOUNG DIAGRAM $\boxplus$, BY WHICH I DENOTE [illegible]

THIS IS OF INTEREST TO US, SINCE IF WE LOOK AT ~~THE~~ OUR FORMULAS FOR $\sigma(T_{\gamma_1}), \sigma(T_{\gamma_2})$ AND PROJECT THEM TO $\int^4 H$ (I.E. TREAT THEM AS POLYNOMIALS, WE SEE IMMEDIATELY THAT THEY GO TO C IN OTHER WORDS:

$$\boxed{T = \operatorname{Im}\sigma \subset \boxplus \oplus Q}$$

IT IS OUR AIM TO SHOW NOW THAT THE $\subset$ IS ACTUALLY AN EQUALITY:

$$\text{Thm: } T = \operatorname{Im}\sigma = \boxplus \oplus Q \text{, and so also}$$

$$T^{*}\otimes Q \cong \Lambda \otimes Q \cong (\boxplus \oplus Q)^{*} \simeq \boxplus \oplus Q$$

THIS ~~IS~~ ~~IS~~ OUR MAIN RESULT QUOTED AT THE BEGINNING, SINCE THE CLASSICAL FORMULA FOR $\dim \boxplus(V)$ WHEN $\dim V = n$ IS $\frac{n^2(n^2-1)}{12}$, AND IN OUR CASE $V = H$ HAS $n = 2g$, WE GET ~~RANK~~ Λ $= \frac{g^2(4g^2-1)}{3} + 1$, AS STATED IN THE FORWARD.

IT REMAINS TO PROVE THE THEOREM. TO THIS END WE MUST ~~LOOK~~ ~~AT SOME DEPTH~~ LOOK AT HOW $\boxplus$, WHICH IS GL-IRREDUCIBLE, DECOMPOSES OVER THE SYMPLECTIC GROUP Sp.

WE HAVE SAID THIS ALREADY

⑮ WE HAVE NOTED THAT Sp ACTS AFFINE LINEARLY ON $\mathcal{L}$. THIS IMPLIES THAT Sp ACTS LINEARLY ON $\operatorname{LIN}(\mathcal{L}, Q)$ AND $\operatorname{QUAD}(\mathcal{L}, Q)$ AND IT IS EASY TO SEE THAT IT PRESERVES HOMOGENEITY.

ALSO, $T = \frac{\mathcal{G}}{C}$ AND WE HAVE SEEN THAT $C \supset [\mathcal{L}, \mathcal{G}]$; HENCE CONJUGATION BY TORELLI INDUCES NO CHANGE ON T, I.E. T IS ALSO AN Sp-MODULE, AND THIS ACTION IS PRESERVED BY THE IDENTIFICATION σ.

THUS, $T = \operatorname{Im}\sigma$ IS AN Sp-SUBMODULE OF $\boxplus \oplus Q$.

EVERYBODY KNOW MAP $R_1 \to R_2 = 0$ IF R_i DIFFERENT IRREPS ? (KEY, IN OUR SUBBOXES)

PROVE THE SEPARATION LEMMA HERE. IF $r \in R$ IRREDUCIBLE, $r \neq 0$ & $s \in S$ SEPARATELY, THEN $r \otimes s$ IN $R \otimes S$ SEPARATELY

⑬

PROOF: LET IT GEN P; P PROJECTS ONTO BOTH R, S. LET $K = \ker(\text{PROJ}_R) = \text{all } (0, \Delta) \text{ in } P$ ... $R \otimes S$ ... $K \subset S$!

COR: ~~[illegible]~~ GET WELL DEFINED MAP $R \to S/K$ BY: LIFT r TO P & PROJECT TO S/K. THIS MAP IS STRUCTURE ... IT $= 0$ SINCE R NOT IN S/K: SO $S/K = 0$ $K = S$.

COR2: IF S IS ALSO IRREP $R \not\cong S$ [illegible]. $r \otimes s$ GEN — IN $R \otimes S$ PROJECTS NON TRIV TO R, S IT GEN $R \otimes S$

IF $P \subset R \oplus S$ PROJECTS ONTO R AND S THEN $= R \oplus S$

SHOULD PROVE IT IN THIS FORM:

IF $P \subset R \oplus S$ PROJECTS ONTO R AND S, AND R, S HAVE NO COMMON IRREPS THEN $P = R \oplus S$

THEN DO COROLLS:

A) IF $x \in R \oplus S$ PROJECTS TO GEN OF R, S THEN x GEN $R \oplus S$

— DITTO $\bigoplus R_i$ NO COMMON IRREPS.

B) IF R_i IRREPS & PROJ FOR $R \neq 0$ & PROJECTIONS ONTO THEN x GENS

C) IF R_i S ... & x PROJECTS NON 0 WHERE THEN x GENS

D) JUST INTUITION IF $R_i S_i$ INDEP TO KNOW x GENERATES! SUFF TO KNOW $\neq 0$

THIS WILL GREATLY AID US IN DETERMINING ~~[illegible]~~ T: ~~[illegible]~~

~~A DIRECT SUM OF THE Sp IRREDUCIBLE~~ OUR SECOND CLASSICAL RESULT IS THAT THE GL IRREDUCIBLES REDUCE COMPLETELY OVER O OR Sp BY THE PROCESS OF <u>CONTRACTION</u>, WHICH WE NOW EXPLAIN.

GIVEN ~~A~~ BILINEAR FORM B ON V AND $1 \leq i \neq j \leq n$, WE DEFINE A MAP $C_{ij} : V^{\otimes n} \to V^{\otimes n-2}$ BY

$$C_{ij}\left(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_j \otimes \cdots \otimes x_n\right) = B(x_i, x_j)\, \underbrace{x_1 \otimes \cdots \otimes x_n}_{\text{LEAVE OUT } x_i, x_j \text{ FACTORS}}$$

NOTE THAT IF B IS $\left\{\begin{matrix}\text{SYMMETRIC}\\ \text{ANTISYMMETRIC}\end{matrix}\right\}$ THEN $C_{ji} = \left\{\begin{matrix}C_{ij}\\ -C_{ij}\end{matrix}\right\}$

ALSO, IF $\Theta \in V^{\otimes n}$ IS $\left\{\begin{matrix}\text{SYM}\\ \text{ANTISYM}\end{matrix}\right\}$ ON INDICES $i\ j$, THEN $C_{ij}(\Theta) = \left\{\begin{matrix}C_{ji}(\Theta)\\ -C_{ji}(\Theta)\end{matrix}\right\}$

~~[illegible]~~ SO IF $\Theta \in V^{\otimes n}$ HAS THE <u>OPPOSITE</u> SYMMETRY ON INDICES $i\ j$ FROM BILIN FORM THEN $C_{ij}(\Theta) = 0$.

[NOTE: IF $G \subset GL$ LEAVES B INVARIANT, THEN THE CONTRACTION MAPS ARE <u>G-HOMS</u>! SO $\ker, \operatorname{Im} C_{ij}$ ARE G-MODS

AS AN EXAMPLE, CONSIDER THE SUBSPACE $\boxplus \subset H^{\otimes 4}$. FOR $\Theta \in \boxplus$

~~[illegible]~~ WE HAVE SYMMETRIES (12) (34) $(13)(24)$ ~~[illegible]~~

~~[illegible]~~ OUR BILINEAR FORM IS THE (ANTISYMMETRIC) INTERSECTION FORM J.

HENCE WE GET:

$$C_{12} = C_{34} = 0$$

$$C_{13} = C_{14} = C_{24} = C_{23}$$

SO THERE IS ONLY ONE CONTRACTION POSSIBLE, I.E. C_{13}, FOR $\boxplus$ ON H WITH ITS NATURAL BILIN ARISING FROM $H^{\otimes n}$

THE CLASSICAL REDUCTION THEOREM FOR ~~[illegible]~~ GL IRREDUCIBLES ON A SPACE WITH $\left\{\begin{matrix}\text{SYM}\\ \text{ANTISYM}\end{matrix}\right\}$ ~~[illegible]~~ BILIN FORM ~~[illegible]~~ SAYS THAT IT "DECOMPOSES ~~[illegible]~~ BY MEANS OF CONTRACTIONS" IN OUR CASE, WITH ONLY ONE CONTRACTION WE GET:

$$W = \ker C_{13} : \boxplus \to H \otimes H$$ IS IRREDUCIBLE (THIS IS THE <u>WEYL</u> TENSOR, SYMPLECTIC FORM

$\boxplus = \ker C_{13} \oplus$ ~~[illegible]~~ C_{13} OVER Q ~~[illegible]~~ C_{13} MAY DECOMPOSE FURTHER.

OUR FIRST AIM IS TO DECOMPOSE $\boxplus$ COMPLETELY INTO IRREDUCIBLES. WE WILL FIND: $\Lambda^2 = J_0 \oplus Q$

~~[illegible]~~ $C_{13} \cong \Lambda^2$, AND: $\boxplus = W \oplus J_0 \oplus Q$ ~~[illegible]~~, WHERE J_0, RECALL, WAS $\ker(J : \Lambda^2 \to Q)$ AND IS, WE WILL SEE, IRREDUCIBLE ~~[illegible]~~

THIS WILL SHOW THAT [illegible]. THEN WE WILL SHOW THAT IMAGE PROJECTS ONTO $W \oplus J_0 \oplus Q \oplus Q$ AND THIS

⑯ WE FIRST SHOW HOW THE REDUCTION THEOREM WORKS IN A SIMPLE BUT USEFUL CASE, NAMELY $\Lambda^2 \subset H\otimes H$. IN THIS CASE THERE IS ONLY ONE CONTRACTION $C_{12}:\Lambda^2\to Q$ AND $C_{12}(x\wedge y) = C_{12}\left(\frac{x\otimes y - y\otimes x}{2}\right) = \frac{x\cdot y - y\cdot x}{2} = x\cdot y$

THUS C_{12} IS THE SAME AS OUR MAP J, THE INTERSECTION FORM ON Λ^2. THE KERNEL OF C_{12} IS WHAT WE HAVE BEEN CALLING J_0, AND J_0 IS THEN IRREDUCIBLE OVER Sp. TO SEE THE SPLITTING OF Λ^2 INTO $J_0\oplus Q$, NOTE THAT THE ELEMENT $\theta = \sum_{i=1}^{g} a_i\wedge b_i$ FOR a_i, b_i A SYMPLECTIC BASIS OF H, IS AN INVARIANT OF Sp IN Λ^2, AND THAT $J\left(\frac{1}{g}\theta\right) = 1$ IN Q. HENCE Λ^2 SPLITS AS $J_0\oplus(\theta)$, WHERE (θ) IS THE ONE DIMENSIONAL INVARIANT SUBSPACE GENERATED BY θ, AS A REPRESENTATION AREA (θ) IS TRIVIAL, I.E. $\simeq Q$.

AS A COROLLARY, WE HAVE:

IF $a\cdot b = 1$ IN H THEN (FOR $g\geq 2$) $a\wedge b$ GENERATES $\Lambda^2 H$ OVER Sp.

PROOF: $J(a\wedge b) = 1$ SO $a\wedge b \notin J_0$. ALSO, FOR $g\geq 2$, $a\wedge b\notin(\theta)$.

~~TH~~ HENCE $a\wedge b$ PROJECTS NONTRIVIALLY INTO BOTH SUMMANDS J_0 AND (θ), SO BY SEPARATION LEMMA OR ITS COROLLARY, IT GENS ALL OF $\Lambda^2 H$

~~AND SO THE Sp SUBMODULE IT GENERATES CONTAINS BOTH J_0 AND (θ), I.E. IS ALL OF~~ $\Lambda^2 H$

NOTE: THIS COROLLARY CAN BE PROVED DIRECTLY ALSO BY SIMPLY MOVING $a\wedge b$ AROUND BY Sp AND GETTING A BASIS FOR Λ^2, (AS IN CUSP FORMS + BC VOL

⑰ NOW WE LOOK AT THE REDUCTION OF ⊞ USING THE CONTRACTION $C_{13}: \boxplus \to H\otimes H$. WE SHOW FIRST THAT $\operatorname{Im} C_{13} = \Lambda^2 H$, AND HENCE BY THE REDUCTION THEOREM $\boxplus \underset{Sp}{\cong} W\oplus\Lambda^2 H$ WHERE $W = \ker C_{13}$ (NOTE: W IS THE SYMPLECTIC ANALOGUE OF THE SO CALLED WEYL TENSOR) ~~[illegible]~~. THIS GIVES US THEN THE COMPLETE REDUCTION OF ⊞ INTO IRREDUCIBLES: $\boxplus \underset{Sp}{\cong} W\oplus J_0\oplus Q$.

ACTUALLY, WE CAN SHOW THAT $C_{13}\left(S^2(H\otimes H)\right) \subset \Lambda^2 H$, SO THIS IS IS CERTAINLY TRUE OF $\boxplus \subset S^2(S^2 H) \subset S^2(H\otimes H)$. TO SEE THIS, NOTE THAT $S^2(H\otimes H)$ IS GENERATED BY ~~[illegible]~~

$(a\otimes b)\circ(c\otimes d) = \frac{1}{2}\Big((a\otimes b)\otimes(c\otimes d) + (c\otimes d)\otimes(a\otimes b)\Big)$. APPLYING C_{13} WE GET $\frac{1}{2}\Big((a\cdot c)(b\otimes d) + (c\cdot a)(d\otimes b)\Big) = (a\cdot c)\,\frac{b\otimes d - d\otimes b}{2} = (a\cdot c)\, b\wedge d$, I.E.,

$$C_{13}\Big((a\otimes b)\circ(c\otimes d)\Big) = (a\cdot c)\, b\wedge d \in \Lambda^2 H.$$

TO SEE THAT $C_{13}(\boxplus)$ IS ACTUALLY EQUAL TO $\Lambda^2 H$, WE APPLY C_{13} TO THE ELEMENT $(a\circ a)\circ(b\circ b) - (a\circ b)\circ(a\circ b)$ WHERE $a\cdot b = 1$ IN H. NOTE THAT THIS IS JUST THE PROJECTION OF $\sigma(T_\gamma) \in \boxplus\oplus Q$ TO $\boxplus$, WHERE γ IS SOME BSSC BOUNDING A GENUS 1 SURFACE WHOSE H_1 IS GEN BY a, b.

WE GET: $a\circ a = a\otimes a \quad b\circ b = b\otimes b \quad a\circ b = \frac{a\otimes b + b\otimes a}{2}$

$$C_{13}\left\{(a\otimes a)\circ(b\otimes b) - \left(\frac{a\otimes b + b\otimes a}{2}\right)\circ\left(\frac{a\otimes b + b\otimes a}{2}\right)\right\}$$

$$= (a\cdot b)\, a\wedge b - \frac{1}{4} C_{13}\left\{(a\otimes b)\circ(a\otimes b) + 2(a\otimes b)\circ(b\otimes a) + (b\otimes a)\circ(b\otimes a)\right\}$$

$$= [illegible]\, a\wedge b - \frac{1}{4}\left\{0 + 2(a\cdot b)(b\wedge a) + 0\right\}$$

$$= \frac{3}{2}\, a\wedge b$$

SINCE $C_{13}(\boxplus)$ IS AN Sp-SUBMODULE OF $\Lambda^2 H$ AND CONTAINS $\frac{3}{2}\, a\wedge b$, BY THE PREVIOUS COROLLARY WE HAVE $C_{13}(\boxplus) = \Lambda^2 H$.

(18) WE PROCEED TO THE PROOF OF THE MAIN THEOREM: ~~[illegible]~~ $= \operatorname{Im}\sigma\otimes Q = \boxplus\oplus Q$

I.E. $\sigma: \mathcal{T} \longrightarrow \boxplus\oplus Q$ IS, WHEN TENSORED WITH Q, ONTO.

SINCE $\boxplus\oplus Q$ SPLITS IRREDUCIBLY INTO $W\oplus J_0 \oplus Q\oplus Q$, IT IS ENOUGH TO SHOW THAT:

a) $\operatorname{Im}\sigma\otimes Q$ PROJECTS ONTO $\begin{smallmatrix}W\\ J_0\end{smallmatrix}$ ~~[illegible]~~

b) $\operatorname{Im}\sigma\otimes Q$ PROJECTS ONTO $Q\oplus Q$ (TRIVIALLY FOR 1, IMσ NOT [illegible]

THE FORMER IMPLIES THAT $\operatorname{Im}\sigma\otimes Q$ CONTAINS W AND J_0, SO COUPLED WITH b) WE HAVE $\operatorname{Im}\sigma\otimes Q$ IS ALL OF $\boxplus\oplus Q$.

WE HAVE ALREADY PROVED b). TO PROVE a): ~~WE HAVE ALREADY SEEN THAT $\sigma(T_\gamma)$~~ PROJECTED TO $\boxplus$ ~~$= \frac{3}{2}(a\wedge b)$~~

~~SO THAT $\operatorname{Im}\sigma$, PROJECTED DOWN TO $\boxplus = W \oplus \Lambda^2 H$, DOES PROJECT ONTO THE $\Lambda^2 H$ FACTOR.~~ IF WE CAN FIND AN ELEMENT OF $\mathcal{J}$ WHICH LANDS IN W, THEN WE WILL HAVE PROVED a). TO DO THIS, WE CONSTRUCT A MAP IN $\mathcal{J}$ WHICH GIVES 0 IN $\Lambda^2 H$, I.E., FOR WHICH $C_{13} = 0$, BUT WHICH $\neq 0$ IN $\boxplus$

LET $f_i = T_{\gamma_i}$ WHERE γ_i ARE GENUS 1 BSCC'S BOUNDING SURFACES S_i WITH HOMOLOGY BASES $[a_1, b_1]$; $[(a_1+b_2), b_1]$; $[(a_2 + b_1), b_2]$; $[a_2, b_2]$

(NOTE: THIS CAN BE DONE IN ANY SURFACE OF GENUS ≥ 2)

COMPUTING $\sigma(f_i)$ WE GET RESPECTIVELY:

$$\sigma(f_1) = \overbrace{(a_1\circ a_1)(b_1\circ b_1) - (a_1\circ b_1)^2}^{\boxplus \text{ PART}} + \overbrace{1}^{\text{CONSTANT: IN } \mathbb{Q}}$$

$$\sigma(f_2) = ((a_1+b_2)\circ(a_1+b_2))(b_1\circ b_1) - (a_1\circ b_1 + b_1\circ b_2)^2 + 1$$

DITTO ABOVE 2 WITH 1'S ↔ 2'S

NOW $\sigma(f_1^{-1} f_2 f_3 f_4^{-1})$ HAS NO $\mathbb{Q}$ PART, SO IS IN $\boxplus$

AND APPLY C_{13} TO IT WE GET

$$C_{13}(\sigma f_2 + \sigma f_3 - \sigma f_1 - \sigma f_4) = \frac{3}{2}\{(a_1 + b_2)b_1 + (a_2 + b_1)b_2 - a_1 b_1 - a$$

$$= 0$$

~~THUS $C_{13} \in \operatorname{Ker}(\boxplus$~~

HENCE $\sigma(f_1^{-1} f_2 f_3 f_4^{-1}) \in \operatorname{Ker} C_{13} = W$. IT REMAINS TO SEE THAT IT IS $\neq$

CALCULATING $\sigma f_2 + \sigma f_3 - \sigma f_1 - \sigma f_4$ WE GET:

$$(a_1\circ a_1 + 2(a_1\circ b_2) + (b_2\circ b_2))(b_1\circ b_1) - (a_1\circ b_1)^2 - 2(a_1\circ b_1)(b_1\circ b_2) - (b_1\circ b_2)^2$$
$$+ (a_2\circ a_2 + 2(a_2\circ b_1) + (b_1\circ b_1))(b_2\circ b_2) - (a_2\circ b_2)^2 - 2(a_2\circ b_2)(b_1\circ b_2) - (b_1\circ b_2)^2$$
$$- (a_1\circ a_1)(b_1\circ b_1) + (a_1\circ b_1)^2$$
$$- (a_2\circ a_2)(b_2\circ b_2) + (a_2\circ b_2)^2$$
$$= 2\{(a_1\circ b_2)(b_1\circ b_1) + (b_1\circ b_1)(b_2\circ b_2) + (a_2\circ b_1)(b_2\circ b_2) - (a_1\circ b_1)(b_1\circ b_2) - (a_2\circ b_2)(b_1\circ b_2) - (b_1\circ b$$

THIS IS NON ZERO IN $\boxplus$ (TO PROVE: EXPAND ALL THE $b_1 b_1 b_2 b_2$ TERMS IN $H^{\otimes 4}$ & SEE THEY DON'T CANCEL.

→ OR DO IT THIS WAY: SWITCHING $a_1 b_1 \leftrightarrow a_2 b_2$ SWITCHES f_1, f_2 AND $f_3 f_4^{-1}$. CHOOSE SUM L = [illegible] $b_1 b_2 + b_2 b_1$: [illegible] UNDER [illegible] NEED ONLY SHOW THAT $\sigma(f_1^{-1} f_2)$ IS NONZERO

THEN EVALUATE DIRECTLY: GET $\sigma(f_1)(L) = ?$ $\sigma(f_2) L = -2$!

THIS COMPLETES THE PROOF OF a). IT REMAINS TO PROVE b), THAT PROJECTING $\mathrm{Im}\,\sigma\otimes\mathbb{Q}\to\mathbb{Q}\oplus\mathbb{Q}$ IS ONTO.

⑲ AS WE HAVE SEEN FROM THE FORMULAS FOR σ ON GENUS 1 AND 2 MAPS, THE PROJECTION OF THESE MAPS TO THE SECOND $\mathbb{Q}$ OF $\mathbb{Q}\oplus\mathbb{Q}$ (I.E. THE $\mathbb{Q}$ OF $\boxplus\otimes\mathbb{Q}$), IS GIVEN BY GENUS 1 $\to$ 1 GENUS 2 $\to$ Q. NOW WE DETERMINE THE PROJECTION TO THE FIRST $\mathbb{Q}$, WHICH IS GIVEN BY $\boxplus\xrightarrow{C_{13}}\Lambda^2 H\xrightarrow{J}\mathbb{Q}$.

WE HAVE ALREADY SEEN THAT A GENUS 1 MAP BOUNDING A SURFACE WITH HOMOLOGY a, b GOES TO $\frac{3}{2}a\wedge b$ IN $\Lambda^2 H$, SO IT GOES TO $\frac{3}{2}$ IN $\mathbb{Q}$. TO SEE THE IMAGE OF A GENUS 2 MAP, E.G.

$$\{(a_1\circ a_1)(b_1\circ b_1)-(a_1\circ b_1)^2\}+\{\text{DITTO } a_2, b_2\}+2\{(a_1\circ a_2)(b_1\circ b_2)-(a_1\circ b_2)(b_1\circ a_2)\}+$$

WE MUST FIRST EVALUATE C_{13} ON THE $\boxplus$ PART (I.E., IGNORE THE "2")

ON THE FIRST TWO BRACKETS WE GET $\frac{3}{2}a_1\wedge b_1+\frac{3}{2}a_2\wedge b_2$

ALSO WE HAVE

$$C_{13}\{(a_1\circ a_2)\circ(b_1\circ b_2)\}=C_{13}\left\{\left(\frac{a_1\otimes a_2+a_2\otimes a_1}{2}\right)\circ\left(\frac{b_1\otimes b_2+b_2\otimes b_1}{2}\right)\right\}$$

$$=\frac{1}{4}\left((a_1\cdot b_1)a_2\wedge b_2+0+0+(a_2\cdot b_2)a_1\wedge b_1\right)=\frac{a_1\wedge b_1+a_2\wedge b_2}{4}$$

AND $$C_{13}\{(a_1\circ b_2)\circ(b_1\circ a_2)\}=C_{13}\left\{\frac{a_1\otimes b_2+b_2\otimes a_1}{2}\circ\left|\frac{b_1\otimes a_2+a_2\otimes b_1}{2}\right|\right\}$$

$$=\frac{1}{4}\left((a_1\cdot b_1)b_2\wedge a_2+0+0+(b_2\cdot a_2)a_1\wedge b_1\right)=\frac{b_2\wedge a_2-a_1\wedge b_1}{4}=-\frac{a_1\wedge b_1+a_2\wedge b_2}{4}$$

SO THE TOTAL RESULT IS

$$\frac{3}{2}(a_1\wedge b_1+a_2\wedge b_2)+2\left(\frac{a_1\wedge b_1+a_2\wedge b_2}{4}-\left(-\frac{a_1\wedge b_1+a_2\wedge b_2}{4}\right)\right),\ \text{I.E.}$$

$$C_{13}\sigma(\text{GENUS 2})=\frac{5}{2}(a_1\wedge b_1+a_2\wedge b_2)$$

APPLYING J TO THIS, WE GET $\sigma(\text{GENUS 2})$ PROJECTS TO $\frac{10}{2}=5$ IN THE FIRST $\mathbb{Q}$.

WE THUS HAVE THE PROJECTIONS OF GENUS 1, 2 INTO $\mathbb{Q}\oplus\mathbb{Q}$ ARE $(\frac{3}{2}, 1)$ AND $(\frac{10}{2}, 2)$ RESPECTIVELY. SINCE $\det\begin{pmatrix}\frac{3}{2} & 1\\ \frac{10}{2} & 2\end{pmatrix}=\frac{1}{2}\begin{vmatrix}3 & 1\\ 10 & 2\end{vmatrix}=-2$, WE SEE THAT THE PROJECTION $\otimes\mathbb{Q}$ IS ALL OF $\mathbb{Q}\oplus\mathbb{Q}$. THIS PROVES b) AND HENCE THE MAIN THEOR

⑳ **REMARK**: (UNPUBLISHED WORK FROM '18) THE PROJECTION OF σ TO $\ydiagram{2,2}$, I.E. THE COMPOSITION MAP

$$\mathcal{J} \xrightarrow{\sigma} \ydiagram{2,2} \oplus Q \xrightarrow{\text{PROJ}} \ydiagram{2,2}$$ HAS A DESCRIPTION

IN TERMS OF THE ACTION OF $\mathcal{J}$ ON $\pi_1 / [\pi_1, [\pi_1, [\pi_1, \pi_1]]]$;

IN FACT: a) $\mathcal{J}$ ACTS TRIVIALLY ON $\pi_1 / \underbrace{[\pi_1, [\pi_1, \pi_1]]}_{\pi_1^{(3)}}$.

b) HENCE FOR $f \in \mathcal{J}$, $x \in \pi_1$, $f(x)x^{-1} \in \pi_1^{(3)}$

AND FOR $x \in \pi_1/\pi_1^{(4)}$, $f(x)x^{-1} \in \pi_1^{(3)}/\pi_1^{(4)}$ WHICH IS <u>ABELIAN</u>, IN FACT A FREE FINITE RANK ABELIAN GROUP

c) THE MAP $x \to f(x)x^{-1}$ IS A <u>HOM</u> $\pi_1 \to \pi_1^{(3)}/\pi_1^{(4)}$ & SO FACTORS THROUGH H, DETERMINING A HOM $\delta f : H \to \pi_1^{(3)}/\pi_1^{(4)}$

d) THE MAP $\delta : \mathcal{J} \to \mathrm{Hom}(H, \pi_1^{(3)}/\pi_1^{(4)})$ IS <u>ALSO</u> A HOMOMORPHISM; $\mathrm{Hom}(H, \pi_1^{(3)}/\pi_1^{(4)}) \cong H^* \otimes \pi_1^{(3)}/\pi_1^{(4)}$ CAN BE IDENTIFIED WITH $H \otimes \pi_1^{(3)}/\pi_1^{(4)}$ AND $\pi_1^{(3)}/\pi_1^{(4)} \otimes 3$ IDENTIFIED WITH $\ydiagram{2,1}$,

SO $H \otimes \pi_1^{(3)}/\pi_1^{(4)}$ IDENTIFIED WITH $\ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1}$

e) THE IMAGE OF δ IS IN $\ydiagram{2,2}$, AND δ(GROUP 1, 2) HAS SAME FORMULAS AS FOR OUR MAP σ

THUS IDENTIFYING δ AND $\mathcal{J} \xrightarrow{\sigma} \ydiagram{2,2} \oplus Q \to \ydiagram{2,2}$.