P.S. This means that the faithfulness of my invariant for 3 braid knots becomes: "if ~~two~~ 3 braids with the same exponent sum define different knots then they have different Alexander polynomials" – do you know if that's true/conceivable?

Friday ①

Dear Joan,

I got your phone call this morning and I thought I would write down that result about normalisation for 3-braids which define knots when you close them.

Accept for the moment that I can prove

$\left(\frac{d}{dt}V_2\right)(1) = 0$ (for all knots, not just 3 braids).

On p. 13 of my other letter you'll find the formula (for 3-braids which are knots)

$$V_2(t) = t^{\frac{e}{2}}\left((1+t+t^2)A(t) + 1 + t^e + t + \frac{1}{t}\right)$$

I say in the letter that $A(t)$ is as it occurs in the determinant of the Burau rep. That's not quite right, it's – it. Let $\Delta(t)$ be $a + b(t+t^{-1}) + c(t^2+t^{-2}) + \ldots$ such that $\Delta(1) = -1$. Then claim $A(t) = t^n\Delta(t)$ with $n = \frac{e}{2} - 1$. I invite you to check an example.

Proof Note first that since $V_2(1) = 1$, $A(1) = -1$.

Now differentiate;

$$0 = V_2'(1) = \frac{e}{2}(3A(1)+4) + e\,3A(1) + 3n\Delta(1) + 3\Delta'(1) + e$$

it's trivial that $\Delta'(1) = 0$ so $0 = \frac{e}{2} - 3 + e - 3n$

$$= \frac{e}{2} - 1 - n \quad \text{so} \quad n = \frac{e}{2} - 1$$

how's that!

This formula reduces the time for my calculations by a huge factor.

Now I've got to prove $V_{\hat{\alpha}}'(1) = 0$. ~~f~~for ~~all~~ knots $\hat{\alpha}$. I believe this is a rather deep property of knots. The proof will be a little tricky though I consider it one of my best. What you need is to find my Hecke algebras inside some fixed algebra so that everything varies with a parameter t. Fortunately this is provided by the Pimsner-Popa-Temperley-Lieb representation – see pp. 6–7 of my "Braid groups – Hecke algebras" preprint or Baxter's book "Exactly solved models in statistical mechanics". To handle the signs I'm going to take a negative square root in the e_i formula. Thus if $g_i = \sqrt{t}(t e_i - (1-e_i))$ then in the model it is the matrix

$$1 \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{\begin{pmatrix} -\sqrt{t} & 0 & 0 & 0 \\ 0 & t^{3/2} - t^{1/2} & -t & 0 \\ 0 & -t & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{t} \end{pmatrix}} \otimes 1 \otimes \cdots$$

occupying the ith & $(i+1)$th slots in some well chosen isomorphism $M_4(\mathbb{C}) \cong M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$

If $\pi_t : \sigma_i \to g_i$ is the representation from B_n then

$$V_{\hat{\alpha}}(t) = \mu^{n-1} \, Tr(h(t)\, \pi_t(\alpha))$$

where Tr (it's unique) is the trace on $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots$ with $Tr(1) = 1$, and $h(t)$ is $\begin{pmatrix} \frac{t}{1+t} & 0 \\ 0 & \frac{1}{1+t} \end{pmatrix} \otimes \begin{pmatrix} \frac{t}{1+t} & 0 \\ 0 & \frac{1}{1+t} \end{pmatrix} \otimes \cdots\cdots$. We must differentiate this expression and put $t = 1$.

The main thing you have to know to do the computation is that, when $t=1$, π factors through the symmetric group and conjugation by $\pi(\alpha)$ just permutes the tensor product components in the obvious way. So now let's go

$$V'_{\alpha}(t) = (n-1)(\mu)^{n-2}\mu' \,Tr(\quad) + \mu^{n-1}\,Tr(h'(t)\,\pi_t(\alpha)) + \mu^{n-1}\,Tr(h(t)\,\pi'_t(\alpha))$$

It's trivial that $\mu'=0$ so the first term goes. ~~The se~~ So I must show that the other terms go.
The second is easiest so I'll start with that. Call

h the matrix $\frac{1}{4}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Then $h'(1) = h\otimes 1\otimes 1\ldots\otimes 1 + 1\otimes h\otimes 1\ldots + 1\otimes 1\otimes h\otimes 1\ldots$ $+\ldots + 1\otimes 1\otimes 1\ldots\otimes h$. But if $t=1$, $\pi_t(\alpha)$ is an n cycle since α is a knot. So we can shunt the h around to the first term and we only have to show that

$$Tr((h\otimes 1\otimes 1\ldots)\,s_1 s_2\ldots s_{n-1}) = 0 \qquad (s_i = \pi_1(\sigma_i))$$

If e_{ij} are matrix units for $M_2(\mathbb{C})$,

$$h = \tfrac{1}{4}(e_{11} - e_{22})\otimes 1\otimes 1\ldots\,)$$

Expanding $s_1 s_2\ldots s_{n-1}$ we see that the only terms that in the product $(h\otimes 1\otimes 1\ldots)\,s_1 s_2\ldots s_{n-1}$ that can possibly contribute to the trace are $(e_{11}\otimes 1\otimes 1\ldots)(e_{11}\otimes e_{11}\otimes\ldots\otimes e_{11})$ and $(e_{22}\otimes 1\otimes 1\ldots)(e_{22}\otimes e_{22}\otimes\ldots\otimes e_{22})$ and these contribute opposite signs. Hence the second term goes. (If you don't like matrix units, just think of how $h\otimes 1\otimes 1\ldots$ and $s_1 s_2\ldots s_{n-1}$ are acting on $\mathbb{C}^n$).

Now comes the tough one.

$\pi_t'(\alpha)$ will be, by the product rule, a sum of terms of the form $\alpha_i g_i' \beta_i$ and $\alpha_i \gamma_i (g_i^{-1})' \delta_i$ ... $h(t)=1$ so

I only have to show that $\mathrm{Tr}(g_i' \beta)$ and $\mathrm{Tr}((g_i^{-1})' \beta)$ vanish at 1

differentiating g_i we obtain, at $t=1$,

$$g_i' = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$ and since $s_i = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$,

$$g_i' = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} s_i$$

The terms that are going to cancel are the two $\frac{1}{2}$'s with the off diagonal -1

in matrix units $g_i' = \Big[1 \otimes 1 \otimes \otimes \big(\frac{1}{2} e_{11} \otimes e_{11} + e_{11} \otimes e_{22} + e_{22} \otimes e_{11} - e_{12} \otimes e_{21} + \frac{1}{2} e_{22} \otimes e_{22}\big) \otimes 1 \otimes 1 \cdots\Big] s_i$

Since s_i is still in the picture, it will conspire with all the other s_j's to give an n-cycle. So by conjugating by the appropriate permutation, $g_i' \beta$ will be

$$\Big(\frac{1}{2} e_{11} \otimes 1 \otimes \cdots \otimes e_{11} \otimes 1 \cdots + \underbrace{e_{11} \otimes 1 \otimes \cdots \otimes e_{22}}_{ⓐ} + \underbrace{e_{22} \otimes 1 \otimes \cdots \otimes e_{11} \otimes 1}_{ⓑ} \cdots - e_{12} \otimes 1 \otimes \cdots \otimes e_{21} \otimes \cdots + \frac{1}{2} e_{22} \otimes \cdots \otimes e_{22} \otimes \cdots\Big) s_1 s_2 \cdots s_{n-1}$$

same position

the adjacency of the e_{11}'s will be ruined by the conjugation but we may assume that one occurs in the first slot by shunting

But, as before, terms ⓐ and ⓑ don't match up with anything diagonal in $s_1 s_2 \cdots s_{n-1}$. The terms $\frac{1}{2} e_{11} \otimes 1 \cdots \otimes e_{11}$ and $\frac{1}{2} e_{22} \otimes \cdots \otimes e_{22} \otimes \cdots$ will contribute $\frac{1}{2}$ each and the

$-e_{12} \otimes 1 \otimes \cdots \otimes e_{21} \otimes 1 \otimes 1 \cdots$ will match up precisely same slot as in previous expression

with the term

$$(e_{21}\otimes e_{12}\otimes 1\otimes 1\cdots)(1\otimes e_{21}\otimes e_{12}\otimes 1\cdots)(1\otimes 1\otimes e_{21}\otimes e_{12}\otimes)\cdots(1\otimes 1\cdots\otimes e_{21}\otimes e_{12}\otimes 1\cdots)$$
$$(1\otimes\cdots\otimes 1\otimes e_{22}\otimes e_{22}\otimes 1\cdots)$$
$$(1\otimes 1\cdots\otimes e_{22}\otimes e_{22}\otimes 1\otimes\cdots)$$

same slot

$$= e_{21}\otimes e_{11}\otimes e_{11}\cdots\otimes e_{11}\otimes e_{12}\otimes e_{22}\cdots\otimes e_{22}$$

to give a contribution of -1 to the trace. Thus the trace vanishes.

For g_i^{-1} the argument is the same:

$$g_i^{-1} = 1\otimes\cdots\otimes\begin{pmatrix} -\frac{1}{\sqrt{t}} & & & \\ & 0 & -\frac{1}{t} & 0 \\ 0 & -\frac{1}{t} & t^{-\frac{3}{2}}-t^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{t}} \end{pmatrix}$$

so in fact $(g_i^{-1})'$ at $t=1$ $= \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$

so $(g_i^{-1})'$ at $t=1$ $= \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} S_i$

and the rest goes as before.

Thus $V_L'(1) = 0$.

This is of course not too polished and would probably gain, as would the whole business of my algebras, from a good understanding of the action on $\mathbb{C}^{2n}$. But I love matrix units.

By the way I did more calculations it turns out that for the knots 10_{155} and 8_7 (both 3 braids), the W invariants differ only by a sign. What this means I don't know.

see you end of June
all the best, Vaughan