31 May 1984
Metuchen.

Dear Joan,

First of all my deepest thanks for putting me on to this. None of it would have begun had it not been for our seemingly unproductive first meeting.

Let me begin by summarizing what we need from operator algebras (everything could be phrased without it but it would be unnatural)

<u>Theorem</u> For every $t > 0$ and $t = e^{\frac{2\pi i}{n}}$, $n = 3, 4, 5, \ldots$

there is an algebra A_n (with identity of operators on a Hilbert space or just an abstract $*$-algebra) over $\mathbb{C}$ (this can be dropped to $\mathbb{Z}(t, t^{-1}, \frac{1}{(1+t)^2})$ but let me stick to what I know) generated by projections e_i with $e_i = e_i^*$, $e_i^2 = e_i$ $\quad i = 1, 2, \ldots$

a) $e_i e_{i\pm1} e_i = \dfrac{t}{(1+t)^2} e_i$

b) $e_i e_j = e_j e_i$ if $|i-j| \geq 2$

Together with a trace $tr : A \to \mathbb{C}$ satisfying $tr(a^*) = \overline{tr(a)}$, $tr(1) = 1$

(i) $tr(a^* a) > 0$ if $a \neq 0$ (ii) $tr(ab) = tr(ba)$

c) $tr(w e_{i+1}) = \dfrac{t}{(1+t)^2} tr(w)$ if w is a word on $1, e_1, e_2, \ldots, e_i$

Notes a) these are the only values of $t \in \mathbb{C}$ for which such an algebra is possible (t^{-1} gives the same algebra as t so these are also possible - that's it)

b) Maybe one can handle amphicheirality using a conjugate linear trace - I'm working on it.

c) these conditions are enough to calculate the trace of any word on the e_i's. I show in my Inventiones paper that, if one allows cyclic permutations as well as a), b), $e_i^2 = e_i$, then any word is equivalent to a decreasing word $e_{i_1} e_{i_2} \cdots e_{i_k}$ (i_j's are decreasing)

eg. if you want

$$tr(e_4 e_2 e_1 e_3 e_2 e_4 e_5 e_4)$$

$$\downarrow$$

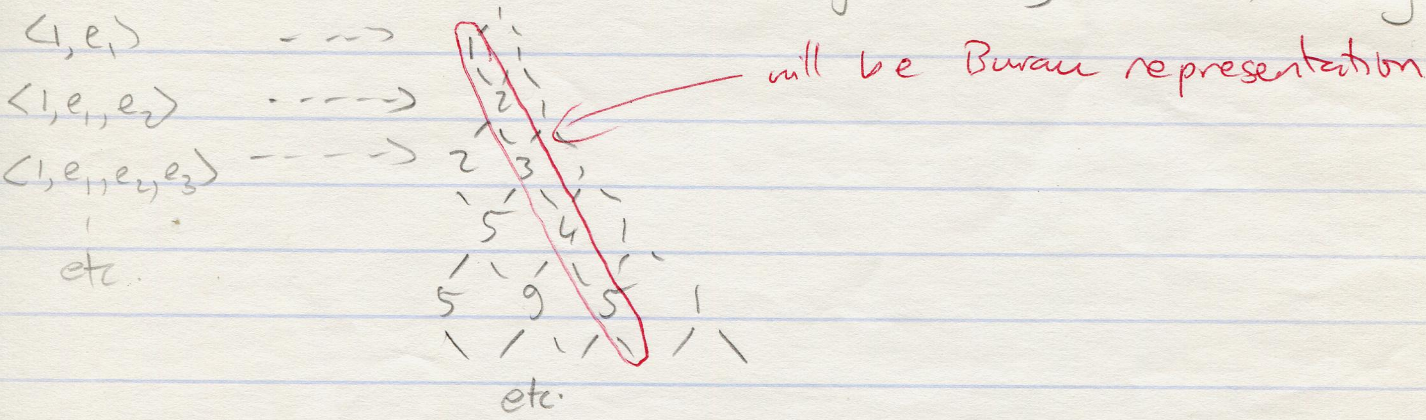
$$e_2 e_1 e_3 e_2 e_4 e_5 e_4 \rightarrow \frac{t}{(1+t)^2} e_2 e_1 e_3 e_2 e_4$$

$$\longrightarrow \frac{t}{(1+t)^2} e_1 e_3 e_2 e_4 \rightarrow \frac{t}{(1+t)^2} e_4 e_3 e_2 e_1 = \left(\frac{t}{(1+t)^2}\right)^5.$$

This means that any representation in which it's convenient to compute the trace can be used [eg. this will imply that for a (proper) knot $\hat{\alpha}$, $V_{\hat{\alpha}}(e^{\frac{2\pi i}{3}}) = 1$, $V_{\hat{\alpha}}(1) = 1$ and in general for a link $V_{\hat{\alpha}}(e^{\frac{2\pi i}{3}})$ = signature of permutation, $V_{\hat{\alpha}}(1)$ is given by (and gives) the number of components]

I have completely analyzed the structure of these algebras in my Inventiones paper and established a notation in my "Braid groups, Hecke algebras ...." preprint

For $t \in \mathbb{R}^+$ the structure is given by the following

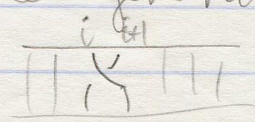


I think you already understand the meaning of this diagram. For all the other algebras see the preprint. If there's a notation in this letter that I don't explain it'll come from the preprint

Note: I believe it's possible to develop the following for the pure braid group and get an invariant in several variables (like Gassner rep). At this stage I don't know how to do it I only remember thinking it out vaguely at one stage.

Note. One can also get the traces on the central projections which will ~~madke~~ make it easy to calculate ~~tr~~ $tr(\Delta^{2n})$ – probably also $tr(\Delta^n)$ but I still don't get one thing for this – it won't be hard.

Now lets turn to the braid group. One must fix first of all a choice of the generators to be consistent with geometry throughout: σ_i =



P

Proposition: There's a representation of B_n in $\langle 1, e_1, \ldots, e_{n-1}\rangle$ which is unique up to scalar multiples given by sending σ_i to $\sqrt{t}\,(t e_i - (1-e_i)) = \sqrt{t}((t+1)e_i - 1)$
let me call this representation π, let $g_i = \pi(\sigma_i)$

Note 1) the normalization is unique also if you want the rest to work - you'll see this if you do the next few calculations
2) For knots the half integer powers of t will drop out in V_L - not for links in general

Strong conjecture π is faithful. (say for $t \in \mathbb{R}$, transcendental)
I'm sure this will fall out of the Garside solution to the conjugacy problem

Amazing observation If $\mu = -\dfrac{(t+1)}{\sqrt{t}}$ and $\alpha \in B_n$

then $\mu^{n-1}\, tr(\pi(\alpha))$ is unchanged by Markov moves.

Proof If $w \in \langle 1, e_1, \ldots, e_{n-1}\rangle$

(i) $tr(g^{-1} w g) = tr(w)$ for $g \in \pi(B_n)$ - trivial

(ii)
$$
\begin{aligned}
tr(w g_n) &= tr(w[\sqrt{t}((t+1)e_{n} - 1)]) \\
&= \sqrt{t}\{tr(w(t+1)e_{n}) - tr(w)\} \\
&= \sqrt{t}\left[(t+1)\frac{t}{(t+1)^2}\, tr(w) - tr(w)\right] \quad \text{by c) on p 1} \\
&= \sqrt{t}\, tr(w)\left[\frac{t}{t+1} - 1\right] \\
&= \sqrt{t}\, tr(w)\left[-\frac{1}{1+t}\right] \\
&= \frac{1}{\mu}\, tr(w)
\end{aligned}
$$

similarly
$$
\begin{aligned}
tr(w g_n^{-1}) &= \frac{1}{\sqrt{t}}\, tr(w)\left[\frac{t+1}{t}\,\frac{t}{(1+t)^2} - 1\right] \\
&= \frac{1}{\sqrt{t}}\, tr(w)\left[\frac{1}{1+t} - 1\right] = \frac{-\sqrt{t}}{1+t}\, tr(w) = \frac{1}{\mu}\, tr(w)
\end{aligned}
$$

Thus when you go from B_{n+1} to B_{n+1} by a Markov move you divide by μ & when you go from B_{n+1} to B_n you multiply by μ. This effect is exactly counterbalanced by dividing by μ^{n-1} initially. - if you have any doubts just check that

$$\mu^{n-1}\,tr(\sigma_1\sigma_2\ldots\sigma_{n-1}) = \mu^{m-1}\,tr(\sigma_1\sigma_2\ldots\sigma_{m-1}) = 1$$

So <u>definition</u> $V_{\hat{\alpha}}(t) = \mu^{n-1}\,tr(\pi(\alpha))$, $\alpha \in B_n$

If Markov's theorem is correct, this is an invariant of the oriented link $\hat{\alpha}$. Let me proceed assuming that the theorem is correct. (I must confess to some vanity in the choice of V - it could also be for von Neumann)

Immediate facts

<u>Lemma 1</u> $V_{\hat{\alpha}\#\hat{\beta}}(t) = V_{\hat{\alpha}}(t)\,V_{\hat{\beta}}(t)$ (an easy exercise)

<u>Lemma 2</u> $V_{\widehat{\alpha^{-1}}}(t) = V_{\hat{\alpha}}(t^{-1})$, (" " ")

<u>Corollary 3</u> if $V_{\hat{\alpha}}(t)$ is not symmetric under $t \to t^{-1}$ then the link is not equivalent to it's mirror image.

This is not right yet

<u>Lemma ~~4~~</u> $V_{\hat{\alpha}}(e^{\frac{2\pi i}{3}}) =$ ~~sign of the permutation defining α~~

<u>Proof</u> Choose $e_i = 1$, $t/(1+t)^2 = 1$, $g_i = -1$ $\mu = -1$ so

$V_{\hat{\alpha}}(e^{\frac{2\pi i}{3}}) = (-1)^{n-1}(e^{\frac{2\pi i}{3}})^{(\text{exponent sum}(\alpha))}$

<u>Lemma 5</u> If $\hat{\alpha}$ is trivial knot $V_{\hat{\alpha}}(t) \equiv 1$ (done above)

<u>Lemma 6</u> $V_{\hat{\alpha}}(1) = (-2)^{n-1-(\text{exponent sum of permutation defined by }\alpha)}$

<u>Proof</u> if $t=1$, $g_i^2 = 1$ so π factors through S_n.

<u>Corollary 7</u> For a proper knot $\hat{\alpha}$, $V_{\hat{\alpha}}(1) = 1$

<u>Calculation of $V_{\hat{\alpha}}(i)$</u>

see "Braid gps. Hecke alg." p. 23

If $t = i$, $\mu = -\sqrt{2}$, $g_k^4 = 1$ and $g_k g_{k\pm1}^2 g_k^{-1} = g_k^2 g_{k\pm1}^2$

$g_k^2 g_{k+1}^2 = -g_{k+1}^2 g_k^2$

These commutation relations allow one to reduce $\pi(\alpha)$ to a word like

$$\pm\, g_1^{n_1} g_2^{n_2} \cdots g_k^{n_k} \quad \text{where } n_i \in \{0, 1, 2, 3\}$$

The trace is then the product of the traces where, if you want to work an example,

$$tr(g_1) = -\frac{1}{\mu} = -\frac{1}{\sqrt{2}}$$

$$tr(g_1^2) = 0$$

$$tr(g_1^3) = -tr(g_1^{-1}) = +\frac{1}{\sqrt{2}}$$

So $V_{\hat{\alpha}}(i) \in \mathbb{Q}(\sqrt{2})$, if $\hat{\alpha}$ is a knot, $V_{\hat{\alpha}}(i) = \pm 1$

The question of whether it's $+1$ or -1 seems interesting - it can be both

for the knot you gave me, $\sigma_1^{-2}\sigma_2\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_1^3\sigma_2^{-1}\sigma_3^{-1}\sigma_2$ a good exercise - its $V_{\hat{\alpha}}(i) = +1$

<u>Calculation of $V_{\hat{\alpha}}(e^{\frac{\pi i}{3}})$</u> If $t = e^{\frac{\pi i}{3}}$, $\mu = -\sqrt{3}$ $g_k^3 = i1$. So the word α can be reduced considerably. If $\alpha \in B_5$ then by Coxeter's criterion one can give a finite list of all possible words α after reduction (up to powers of i). This is an exteremely interesting value of $V_{\hat{\alpha}}$ as $\pi(B_n)$ is always a finite group - eg. $n=5$ it is the group of order 155,520. For higher n it is apparently a semidirect product of a symplectic group over $\mathbb{Z}_3$ by an extraspecial group. Actually there's probably an extra relation around like for $t=i$ which gives a

simple formula – I don't know it.

Note $V_{\hat{L}}(e^{\frac{\pi i}{3}})$ is not necessarily real – see trefoil ($\sqrt{3}\,i$) and as soon as it's not real, the knot is not equivalent to its mirror image.

For B_3 one can also calculate $V_{\hat{L}}(e^{\frac{\pi i}{5}})$ by a similar method since here $\pi(B_3)$ is finite (its probably related to the result of Magnus & Peluso in "On knot groups".

Examples

knots

Trefoil	one handed	$V_{\hat{L}}(t) = t + t^3 - t^4$
	other handed	$V_{\hat{L}}(t^{-1}) = \frac{1}{t} + \frac{1}{t^3} - \frac{1}{t^4}$
$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$	Figure 8	$= 1 - t - t^{-1} + t^{-2} + t^2$
$\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^3$	6_2	$= -1 + 2t + t^{-1} - 2t^2 + 2t^3 - 2t^4 + t^5$
$\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^2\sigma_1^{-2}$	8_{17}	$7 - 6t - 6t^{-1} + 5t^2 + 5t^{-2} - 3t^3 - 3t^{-3} + t^4 + t^{-4}$
		(nothing about reversibility yet)
$(\sigma_1\sigma_2^{-1})^4$	8_{18}	$9 - 7t - 7t^{-1} + 6t^2 + 6t^{-2} - 4t^3 - 4t^{-3} + t^4 + t^{-4}$
$\sigma_1^{-1}\sigma_2^2\sigma_1^{-2}\sigma_2$	6_3	$3 - 2t^{-1} - 2t + 2t^2 + 2t^{-2} - t^3 - t^{-3}$

links

σ_1^2 $\quad -\sqrt{t}(t^2+1)$

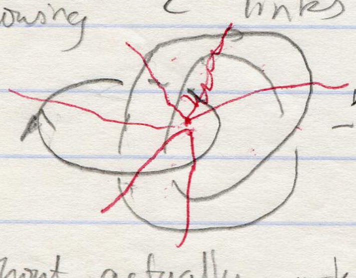
σ_1^{-2} $\quad -t^{-3/2}(t^2+1)$

so different as oriented links!

can detect difference easily at $e^{\pi i/3}$

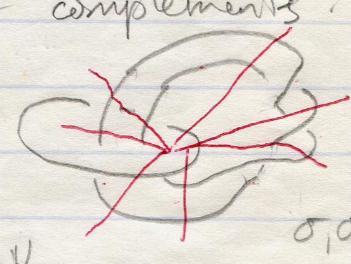
The following 2 links have homeomorphic complements:

L_1



– braid $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^2$

L_2



$\sigma_1\sigma_2^{-3}\sigma_1\sigma_2^2$

without actually working out the invariant explicitly we can tell that the invariant is different

$$V_{L_1}(t) = \frac{(t+1)^2}{t} t^{\frac{3}{2}} \operatorname{tr}\big(((t+1)e_1 - 1)((t^{-1}+1)e_2 - 1)((t+1)e_1 - 1)((t^2-1)e_2 + 1)\big)$$

as $t \to 0$ this tends to 0 by a simple look at the terms (I've done the calculation ans $= -\sqrt{t}(t^4 + t^2 - t + 1)$)

$$V_{L_2}(t) = \frac{(t+1)^2}{\sqrt{t}} \operatorname{tr}\big(((t+1)e_1 - 1)((t^{-3}+1)e_2 - 1)((t+1)e_1 - 1)((t^2-1)e_2 + 1)\big)$$

as $t \to 0$ this tends to ∞ by a (less) simple look at the terms

Notes 1) The last example shows that it would be extremely desirable to have a simple way of calculating $V_{\hat{\alpha}}(0)$. The trouble is the putting $t=0$ gives $\frac{\infty}{\infty}$ problems. A clever contour integral might do the trick.

2) For all these knots $V_{\hat{\alpha}}$ is a Laurent polynomial. This is always true as can be shown using the Hecke algebra approach. A link's invariant will be either a Laurent polynomial or $t^{\frac{1}{2}}$ times a Laurent polynomial depending on the sign of the permutation.

3) The same kind of analysis as in the last example shows that for the braid

$$\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_1^{3}\sigma_2^{-1}\sigma_3\sigma_2\sigma_1^{-1}$$

which has trivial alexander polynomial, $V_{\hat{\alpha}}$ is far from zero. It grows as $-t^6$ as $t \to \infty$.

4) I haven't yet calculated $V_{\hat{\alpha}}$ for any self-respecting 4-braids. This will be exciting.

Use of $V_{\hat{\alpha}}$ to detect knottedness

a) If $\hat{\alpha}$ is the unknot $V_{\hat{\alpha}}(t) = 1$. The following truly crude analysis gives a way of detecting unknottedness.

Let α^+ = +ve exponent sum
α^- = |−ve " " |

Consider $t < 1$. Since e_i's are orthogonal projections

$\|t e_i - (1-e_i)\| = 1$ so $\|g_i\| = \sqrt{t}$
$\|t^{-1} e_i - (1-e_i)\| = t^{-1}$ so $\|g_i^{-1}\| = t^{-\frac{3}{2}}$

Thus if $\alpha \in B_n$,

$$|V_{\hat{\alpha}}(t)| = \mu^{n-1} |tr(\pi(\alpha))| \leq \mu^{n-1} \|\pi(\alpha)\| \leq \mu^{n-1} t^{\frac{1}{2}\alpha_+} t^{-\frac{3}{2}\alpha_-}$$

as $t \to 0$ $\mu \sim t^{-\frac{1}{2}}$ so the asymptotic behaviour of $|V_{\hat{\alpha}}(t)|$ is

$$|V_{\hat{\alpha}}(t)| \leq t^{\frac{1}{2}(\alpha_+ - 3\alpha^- - n + 1)}$$

so provided $\alpha_+ - 3\alpha^- - n + 1 > 0$, $V_{\hat{\alpha}}$ has a zero at zero and $\hat{\alpha}$ is not the unknot.

Note: The same result $\Rightarrow$ if $\alpha_+ - 3\alpha^- - n + 1 > 0$, $\hat{\alpha}$ is not equivalent to its mirror image.

b) If any of the special values $V_{\hat{\alpha}}(i$ or $e^{\frac{2\pi i}{3}}$ or 1 or $e^{\frac{\pi i}{3}})$ is different from 1 then $\hat{\alpha}$ is not the unknot. I can't see any general statement that this implies that is not trivial anyway but it could be extremely useful in particular cases

Detecting the trivial link

<u>Proposition 8</u> $V_{\hat{\alpha}}(t) = \mu^{n-1}$ for $t = e^{\frac{2\pi i}{n}}$ $n=3,4,\ldots$

iff $\alpha \in \ker \pi$ (remember there's a π for each t)

<u>Proof</u> $V_{\hat{\alpha}}(t) = \mu^{n-1}\, tr(\pi(\alpha))$

so $V_{\hat{\alpha}}(t) = \mu^{n-1} \Rightarrow tr(\pi(\alpha)) = 1$. But

$\pi(\alpha)$ is unitary so $\pi(\alpha) = 1 \Rightarrow \alpha \in \ker \pi$

(proof that $tr(u) = 1 \Rightarrow u = 1$ for u unitary:

$$tr((u-1)(u^*-1)) = tr(uu^*) - tr(u) - tr(u^*) + 1$$
$$= 1 - 1 - 1 + 1 = 0$$
$$\Rightarrow (u-1)(u^*-1) = 0$$
$$\Rightarrow u = 1$$ – ask any analyst!)

<u>Corollary 9</u> $\ker \pi$ is a union of Markov equivalence classes for $t = e^{\frac{2\pi i}{n}}$

<u>Corollary 10</u> $V_{\hat{\alpha}}(t) = \mu^{n-1}$ for $t \in \mathbb{C}$, t transcendental $\Longleftrightarrow \alpha \in \ker \pi$

<u>Proof</u> Basically this is because we're dealing with rational functions which would have an accumulation point of zeros by prop. 8. Actually to fill in the details requires some work but I can do it.

<u>Corollary 11</u> same as 9 for t transcendental.

<u>Major comment</u> Thus the faithfulness of π is equivalent to: $V_{\hat{\alpha}}(t) \equiv \mu^{n-1} \Rightarrow \hat{\alpha}$ is trivial link

if this can't be done using Garside's solution of the conjugacy problem and my techniques for looking at Δ^2, I'll eat my hat.

Use of $V_{\hat{\alpha}}$ in calculating braid index

Theorem Let $\alpha \in B_n$, $\alpha \in \ker \pi$ for $t = e^{\frac{2\pi i}{k}}$ $k = 4, 5, \ldots$ Then the braid index of α is n.

Proof If $\alpha \in B_m$ $V_{\hat\alpha}(t) = \mu^{m-1}\, tr(\pi(\alpha))$. Now $\pi(\alpha)$ is unitary so $|tr(\pi(\alpha))| < 1$. Hence $|V_{\hat\alpha}(t)| < |\mu|^{m-1}$. $|\mu| = 2\cos\frac{\pi}{k} > 1$ If $\alpha \in \ker \pi$, $V_{\hat\alpha}(t) = \mu^{n}$ so $m \geq n$ Q.E.D.

This of course gives zillions of words for which one can decide the braid index question. The value $t = i$ is not much use since if $\alpha \in \ker \pi$ α is a pure braid. But $e^{\frac{i\pi}{3}}$ is interesting. One could look up generators and relations for the group of order 155,520 to find some interesting examples. If I knew for sure what the groups are for B_n, $n \geq 6$ I would have many more fascinating examples from their presentations. Of course the simplest way to get in the kernel is to be a product of things in the kernel so we get

Corollary If $\alpha \in B_n$ and the g.c.d. of the exponents of α is different from 1 then the braid index of α is n.

Another interesting case is for B_4 and $e^{\frac{\pi i}{5}}$. I'll use this in my next topic.

In fact all one needs to show is that

$V_L(t)$ is not one of the values of $V_{\hat{\beta}}(t)$

for $\beta \in B_m$, $m < n$. If $\pi(B_m)$ is finite for $m<n$ this question will be decidable. This will give many more examples.

<u>Solution of the $U\sigma_n V\sigma_n^{-1}$ question</u>

You ask whether every word in B_{n+1} is conjugate to a word of the form $U\sigma_n V\sigma_n^{-1}$ for $U, V \in B_n$. In fact I can produce vast families which are not even Markov equivalent to such words. In my work I show that there is a map $E: \langle 1, e_1, \ldots, e_{n-1}\rangle \to \langle 1, e_1, \ldots, e_{n-2}\rangle$ with the property $e_n x e_n = E(x) e_n$ for $x \in \langle 1, e_1, \ldots, e_{n-1}\rangle$

The following calculation is trivial but very suggestive: if U and V are as above

$$tr(U\sigma_n V\sigma_n^{-1}) = tr(E(U)E(V))$$

so

$$V_{\widehat{U\sigma_n V\sigma_n^{-1}}}(t) = \mu^{n-1}\, tr(E(U)E(V))$$

but now it's easy as soon as you know a bit about the possible values of $tr(E(U)E(V))$

For instance $\pi(B_3)$ is infinite for $t = e^{\frac{2\pi i}{5}}$ but $\pi(B_2)$ is $\mathbb{Z}_{10}$ so almost all elements of B_3 are not of the form $U\sigma_n V\sigma_n^{-1}$ (note that on an infinite subgroup of a unitary group the trace will take infinitely many distinct values)

A more interesting case (you probably knew this already for B_3) occurs for B_4: I have shown (admittedly by quite a delicate calculation) that for $t = e^{\frac{\pi i}{5}}$, $\pi(B_4)$ is infinite while $\pi(B_3)$ is finite. In fact $\pi(\sigma_1\sigma_2\sigma_3^{-1})$ is of infinite order. By the infiniteness of the values of the trace, almost all powers $(\sigma_1\sigma_2\sigma_3^{-1})^n$ will have braid index 4 and not be expressible as $U\sigma_n V\sigma_n^{-1}$, even after any number of Markov moves.

Closed 3 braids

Knowledge of the decomposition of π and the relationship of the determinant in the Burau representation with the Alexander polynomial gives a simple formula. Here it is, (e = exponent sum of α, ε = signature of permutation)

$$V_{\hat\alpha}(t) = (\sqrt{t})^e \Big[\underbrace{(1+t+t^2)A(t) + 1 + (-t)^e}_{\text{trace Burau matrix}} + \varepsilon\left(t + \tfrac{1}{t}\right) \Big]$$

thus if the exponent sum is fixed, the Burau normalisation of the Alex poly. is a knot invariant & vice versa

Here $A(t)$ is as it occurs in the Burau representation. Knowledge of $A(t)$ in any normalisation, eg. from tables can be converted into an exact formula since the behaviour of $V_{\hat\alpha}(t)$ for t large is easy to determine without knowing it explicitly.

In particular one obtains (for 3 braids)

a) Easy formulae for $A(i)$, $A(e^{\frac{2\pi i}{3}})$, $A(e^{\frac{\pi i}{3}})$, $V_{\hat\alpha}(-1) = A(-1)$ (for proper knots)

b) (For a proper knot) Unless $A(t)$ is of the form

$$\frac{(t^m - t^{m'}) + (t^{3m} - t^{3m'}) + (t^{m+1} - t^{m'+1}) + (t^{m-1} - t^{m'-1})}{(1+t+t^2)(t^{n_1} - \varepsilon t^{n_2})} \quad (\varepsilon \text{ a sign})$$

then e and the normalisation of $A(t)$ in the Burau rep. are knot invariants

There's plenty more but I'm running out of energy so let me just give you the rather unlikely looking formula for $V_\alpha(t)$ for closed 4 braids. This must be pregnant with consequences:
Let ξ be the Burau rep of B_4 $\otimes$ parity and η the map $B_4 \underset{\sigma_1 = \sigma_3}{\rightarrow} B_3 \xrightarrow{\text{Burau} \otimes \text{parity}} 2\times 2$ matrices.

ordinary trace = sum of diagonal

Then

$$V_{\hat\alpha}(t) = -\left(\frac{t+1}{\sqrt{t}}\right)^3 (\sqrt{t})^{\text{exponent sum}} \left[\frac{t^2}{(1+t)^4}\,\mathrm{trace}(\eta(\alpha)) + \frac{t(1+t+t^2)}{(1+t)^4}\,\mathrm{trace}(\xi(\alpha)) + (-1)^{\text{exp sum}}\frac{(1+t+t^2+t^3+t^4)}{(1+t)^4}\right]$$

$$= -\frac{(\sqrt{t})^{e-3}}{(1+t)}\left(t^2\,\mathrm{trace}(\eta(\alpha)) + t(1+t+t^2)\,\mathrm{trace}(\xi(\alpha)) + (-1)^e(1+t+t^2+t^3+t^4)\right)$$

Checks: for $\alpha = \mathrm{id}$ you should get $-\dfrac{(1+t)^3}{(\sqrt{t})^3}$ - you do.

for $\alpha = \sigma_1\sigma_2\sigma_3$ . . 1 you do.

Since even powers of Δ are in the centre it will be easy to calculate torus knots that are 4 braids

eg. $\Delta^2 = \begin{pmatrix} t^6 & 0 \\ 0 & t^6 \end{pmatrix} \oplus \begin{pmatrix} t^4 & 0 & 0 \\ 0 & t^4 & 0 \\ 0 & 0 & t^4 \end{pmatrix} \oplus 1$ so answer

$$= -\frac{(\sqrt{t})^9}{(1+t)}\left(2t^8 + 3t^5(1+t+t^2) + 1+t+t^2+t^3+t^4\right)$$

$$= -(\sqrt{t})^9\left(1+t^2+t^4+2t^5+t^6+2t^7\right)$$

. If I understood Δ a bit better I could get the general formula.

It's clear what the effect of multiplying by Δ^2 is - maybe you can immediately prove faithfulness with Garside's solution

All for now, thanks again, Vaughan.

Further scribblings

For proper knot $\hat{V}_2(1)=1$ $V_2(e^{\pm\frac{2\pi i}{3}})=1$ so one can form the simpler Laurent polynomial

$$W_1(t) = \frac{1-V(t)}{1-t^3}$$

it seems that this factors again by $1+t$ (If this is true I can prove it) so the best would be

$$W(t) = \frac{1-V(t)}{(1+t)(1-t^3)}$$

Then things look really nice: the trivial knot has an invariant of 0, the trefoil knot 1 and going from a knot to its mirror image is

$$W_{\text{mirror image}}(t) = \frac{1}{t^4}\, W\left(\frac{1}{t}\right)$$

This would mean $\hat{V}_2(-1) \equiv 1 \mod 4$ which is true in every example I've looked at. Note that this says something remarkable about the Alexander polyn of a closed 3-braid.

I'm enclosing a table with $\hat{V}_2$ for all the knots I've done so far. $\hat{V}_2'(1)=0$ seems to be true and I have a proof $-\varepsilon$ (which is rather elegant if I say so myself). A look at the table reveals many interesting possibilities - it seems that $W(t)$ is $t^{\text{some power}} \times$ a monic polynomial P with $P(0)=1$. I have no idea of how to prove that.

By the way, in lieu of a preprint I'm going to give this letter to anyone that wants it

Hope to see you at the end of June

Vaughan