14 Nov. 1984

(1)

Dear Joan,

Heres' a quick summary of my thoughts so far on this bridge plat thing

Set, for each odd n, $p_n = e_1 e_3 e_5 \cdots e_n \in A_n$

(A_n as in, say, my ~~Index~~ Braid gps., Hecke algs. paper)

Fact: p_n is a minimal projection in A_n, i.e.

~~i.e.~~ $p_n A_n p_n \subseteq \mathbb{C} p_n$ - proof by induction.

(see JB - last page)

Fact: this means p_n "lives" in one of the simple ~~compo~~ summands of A_n - In fact it's the one on the extreme left of the Bratteli diagram. This is easily proved since this is the only one that gives the right trace.

gen by e_1, e_2, e_3

i.e. α is a word in σ_1, σ_2 not σ_3

Corollary. If $\alpha \in B_4$ the ~~and if~~ $\pi(\alpha) \in A_3$ in obvious way, then $e_1 e_3 \pi(\alpha)_{\Lambda}^{(e_1 e_3)?} = e_1 e_3 \pi(\alpha')_{\Lambda}^{(e_1 e_3)?}$, α' being the image of α under the homomorphism from B_4 to B_3

Here it is ok to omit the factor $e_1 e_3$ because $e_1 e_3$ commutes with the generators of $\ker(B_4 \to B_3)$, so if $x \in \ker$ we have

$(e_1 e_3) x = (e_1 e_3)^2 x = (e_1 e_3) x (e_1 e_3)$

Corollary α is true for all $x \in \ker$ and, however since the kernel is generated by $g_2 g_1 g_3^{-1} g_2^{-1}$ and $g_1 g_3^{-1}$ This is the only non-trivial case

Corollary (α) $e_1 e_3 (g_2 g_1 g_3^{-1} g_2^{-1}) = e_1 e_3$

Definition define $R : A_n \to \mathbb{C}$ by

$$p_n x p_n = R(x) p_n$$

Proposition (i) R is welldefined on $\bigcup_{n=1}^{\infty} A_n$

(ii) $R(x) = \frac{1}{\tau^{\frac{n+1}{2}}} \operatorname{tr}(p_n x)$

(both trivial) Proof of (ii)

Now define a function on $B_{n+1} \to$ Laurent polys. in t by

$$J_\alpha(t) = \left(-(t+1)\right)^{\frac{n+1}{2}} R(\pi(\alpha))$$??

$\left(\frac{-(t+1)}{\sqrt{t}}\right)^{\frac{n+1}{2}}$?? See p5 tr$(p_n\alpha)$

but these are the same mod t^k

Theorem If the plat $\tilde{\alpha}$ associated with α is a knot then $V_{\tilde{\alpha}}(t) = t^{\text{some power}} J_\alpha(t)$

Proof First let's see how J_α varies on double cosets of the group K gen. by

(i) odd g_i's

(ii) $\sigma_{2k}\sigma_{2k-1}\sigma_{2k+1}\sigma_{2k}$

(iii) $\sigma_{2k}\sigma_{2k-1}\sigma_{2k+1}^{-1}\sigma_{2k}^{-1}$ due already for B_4

$p_n g_i = e_1 e_3 \cdots e_n$

(i) Remember $g_i e_i = t e_i$ so if i is odd

$p_n \pi(\alpha\sigma_i) p_n = R(\pi(\alpha\sigma_i)) p_n = p_n \pi(\alpha) g_i p_n \overset{t p_n \pi(\alpha) p_n}{=} t R(\pi(\alpha)) p_n$

(ii) ~~The~~ Almost all the e_i's in p are irrelevant so it suffices to do the computation $e_1e_3 g_2g_1g_3g_2 \pi(\alpha) e_1 e_3$

and I claim $(e_1e_3)g_2g_1g_3g_2 = t \cancel{g_2g_1g_3g_2} e_1e_3$

first of all as you know $(g_2g_1g_3g_2)g_1(g_2g_1g_3g_2)^{-1} = g_3$ & same $3 \to 1$

so the same is true of the e's and $g_2g_1g_3g_2$ commutes with e_1e_3 so

$(e_1e_3)^2 (g_2g_1g_3g_2) =$

$e_1e_3\, g_2g_1g_3g_2 = (e_1e_3 g_2g_1g_3g_2 e_1e_3) =$ const. g e_1e_3 and we

because $p_n\pi(\alpha)p_n = R(\alpha)p_n \;\forall \alpha \in B_{n+1}$

just have to compute the constant

it's easy to see that $\underbrace{\text{it's just}}_{\text{by the Proposition (ii) on bottom of page 1}}$ $\frac{1}{\tau^2}\operatorname{tr}(e_1e_3g_2g_1g_3g_2)$ and because $e_1e_3(g_2g_1g_3g_2) = e_1e_3(g_2g_1g_3^{-1}g_2^{-1})(g_2g_3^2g_2) = e_1e_3(g_2g_3^2g_2)$

③

by ~~remark~~ 1st corollary, this is $\frac{1}{\tau^2}tr(e_1e_3g_2g_1^2g_2)$

only appearance of subscript 3

$= \frac{1}{\tau}tr(e_1g_2g_1^2g_2)$

now $g_1^2 = (t-1)g_1 + t$ so

$= \frac{t-1}{\tau}tr(e_1g_2g_1g_2) + \frac{t}{\tau}tr(e_1g_2^2)$

$= \frac{t-1}{\tau}tr(e_1g_1g_2g_1) + t\,tr(g_2^2)$

$= (t-1)t^2\,tr(g_2) + t\,tr(g_2^2)$

now $g_2^2 = (t^2-1)e_2 + 1$ so $tr(g_2^2) = \frac{(t^2-1)t}{(1+t)^2} + 1 = \frac{t^2-t+1+t}{1+t}$

$= (t-1)t^2\left(\frac{-1}{1+t}\right) + t\cdot\frac{t^3+t^2}{1+t} = \frac{t^2+t}{1+t} = t$

(there may be a much more elegant way to do this)

thus $g_2g_1g_3g_2e_1e_3 = e_1e_3g_2g_1g_3g_2 = t\,e_1e_3$.

(iii) follows from lemma (α) that $g_{2k}g_{2k-1}g_{2k+1}^{-1}g_{2k}^{-1}p_n = p_n$

Thus we see that if $x, y \in K$

$$J_{x\alpha y} = t^{\text{same power}}J_\alpha$$

Now let's look at stabilization: $R(x) = \frac{1}{\tau^{\frac{n+1}{2}}}tr(e_1e_3\cdots e_nx)$

If $x \in A_n$, n odd then ~~the~~ claim. $R(xg_{n+1}) = \frac{-1}{(t+1)}R(x)$

$R(xg_{n+1}^{-1}) = -\frac{t}{1+t}R(x)$

Proof $R(xg_{n+1}) = \frac{1}{\tau^{\frac{n+3}{2}}}tr(p_{n+2}xg_{n+1}) = \frac{1}{\tau^{\frac{n+3}{2}}}tr(e_1e_3\cdots e_ne_{n+2}xg_{n+1}) = \frac{-1}{1+t}R(x)$

only occurrence of e_{n+2} and e_{n+1}

similarly for g_{n+1}^{-1}

$= \frac{1}{\tau^{\frac{n+3}{2}}}tr(p_nx)(e_{n+2}g_{n+1}) = \frac{1}{\tau^{\frac{n+1}{2}}}tr(p_nx)[(1+t)e_{n+1}-1] = \frac{1}{\tau^{n+1}}tr(p_nx)[\tau(1+t)-1]$

$\frac{t}{1+t} - 1 = \frac{-1}{1+t}$

$\alpha \in B_{n+1}$

Thus

$$J_{\alpha\sigma_{n+1}} = (-(t+1))^{\frac{n+3}{2}} R(\pi(\alpha\sigma_{n+1}))$$
$$= (-(t+1))^{\frac{n+3}{2}} \cdot \frac{-1}{(t+1)} R(\pi(\alpha))$$
$$= (-(t+1))^{\frac{n+1}{2}} R(\pi(\alpha))$$
$$= J_\alpha$$

$$\& \quad J_{\alpha\sigma_{n+1}^{-1}} = t J_\alpha$$

We see by your theorem that J_α depends, up to powers of t, only on $\tilde{\alpha}$.

We now want to compute it for a specific isotoped $\tilde{\alpha}$, i.e. a closed braid representation $\beta \in B_m$, some m, with $\hat{\beta} = \tilde{\alpha}$.

As you know, if $\sigma = \sigma_2\sigma_3 \cdots \sigma_{2m-1}\sigma_3\sigma_4 \cdots \sigma_{2m-2} \cdots \sigma_m\sigma_{m+1}$, $\tilde{\alpha} = \widetilde{\sigma\beta\sigma^{-1}} = \hat{\beta}$. The following you also must know.

$$\sigma\sigma_1\sigma^{-1} = \sigma_1^{-1}\sigma_2\sigma_1, \ \sigma\sigma_2\sigma^{-1} = \sigma_3^{-1}\sigma_4\sigma_3, \ \ldots \ \sigma\sigma_{m-1}\sigma^{-1} = \sigma_{2m-3}^{-1}\sigma_{2m-2}\sigma_{2m-3}$$

This means $\sigma e_1 \sigma^{-1} = \sigma_1^{-1} e_2 \sigma_1$ etc.

Now I claim that if $\varphi : UA_n \to \mathbb{C}$ is any linear functional with $\varphi(1) = 1$, $\varphi(x e_{n+1} y) = \tau\varphi(xy)$, $x, y \in A_n$, then $\varphi = tr$.

This is easy by induction and reduced words on the e_i's.

So I claim that φ defined by $\varphi(x) = \frac{1}{\tau^n} tr(e_1 e_3 \cdots e_{2n-1} \sigma x \sigma^{-1})$ $\underline{\underline{x \in A_n}}$ is such a linear functional.

By induction on n. First of all, well definedness is trivial from $\sigma e_i \sigma^{-1}$ formula, ~~since~~ (i.e. φ is defined on UA_n) so we may use induction on n, trivial for A_1

$$\varphi(x e_{n+1} y) = \frac{1}{\tau^{n+1}} tr(e_1 e_3 \cdots e_{2n-1} e_{2n+1} \sigma x \sigma^{-1} \sigma_{2n+1}^{-1} e_{2n+2} \sigma_{2n+1} \sigma y \sigma^{-1})$$

$$= \cancel{\tau} \frac{1}{\tau^n} tr(e_1 e_3 \cdots e_{2n-1} e_{2n+1} \sigma x y \sigma^{-1})$$

$$= \tau \left(\frac{1}{\tau^n} tr(e_1 e_3 \cdots e_{2n-1} \sigma x y \sigma^{-1}) \right)$$

(note ~~$\sigma x y \sigma$~~ $\sigma x \sigma^{-1}, \sigma y \sigma^{-1} \in A_{2n}$)

$$= \tau \varphi(xy)$$

Thus $\varphi = tr$ and if $\beta \in B_m$

$$V_{\hat\beta}(t) \cancel{\varphi(x)} = \left(-\frac{t+1}{\sqrt{t}}\right)^{m-1} tr(\pi(\beta))$$

$$= \left(-\frac{t+1}{\sqrt{t}}\right)^{m-1} \varphi(\pi(\beta))$$

$$= \left(-\frac{t+1}{\sqrt{t}}\right)^{m-1} \frac{1}{\tau^{m-1}} tr(e_1 e_3 \cdots e_{2m-3} \pi(\sigma \beta \sigma^{-1}))$$

$$= \left(-\frac{t+1}{\sqrt{t}}\right)^{m-1} \cdot \frac{1}{\tau^{m-1}} tr(p_{2m-3} \pi(\sigma \beta \sigma^{-1}))$$

$$= \left(-\frac{t+1}{\sqrt{t}}\right)^{m-1} R(\pi(\sigma \beta \sigma^{-1}))$$

$$= \left(-\frac{t+1}{\sqrt{t}}\right)^{m-1} (-(t+1))^{-m+1} J(\sigma \beta \sigma^{-1})$$

$$= t^{\text{some power}} J(\alpha) \text{ since}$$

$\tilde\alpha \underset{\text{isotopy}}{=} \widetilde{\sigma \beta \sigma^{-1}}$.

"n" $= 2m-3$
"$\frac{n+1}{2}$" $= m-1$

Q.E.D.

$\alpha \in B_{2n}$

Corollary If $\alpha \in \ker \pi$ for $t = e^{\frac{2\pi i}{k}}$ $k = 4, 5, \dots$

& $\tilde{\alpha}$ is a knot then

~~braid~~ crookedness $(\tilde{\alpha}) = n$.

{Arveson
Invitation to C^* algebras

?? Proof. $R(x)$ is a "state" in C^*-sense, ie.

$R(x^* x) \geq 0$ so we have Cauchy Schwarz;

$$|R(x^* y)| \leq R(x^* x)^{\frac{1}{2}} R(y^* y)^{\frac{1}{2}}$$

in particular if u is unitary

$$|R(u)| \leq R(u^* u)^{\frac{1}{2}} R(1^* 1)^{\frac{1}{2}} = 1$$

so that if $\beta \in B_{2m}$, $m < n$

$$\left| J_\beta \left(e^{\frac{2\pi i}{k}}\right) \right| \leq \left(2 \cos \frac{\pi}{k}\right)^m$$

But if $\alpha \in \ker \pi$, $\left| J_\alpha \left(e^{\frac{2\pi i}{k}}\right) \right| = \left(2\cos\frac{\pi}{k}\right)^n \gneq \left(2\cos\frac{\pi}{k}\right)^m$

for $m < n$.

QED

Corollary if α = product of σ_i's & GCD of exponents of $\alpha > 1$, then ~~bri~~ crookedness$(\tilde{\alpha}) = n$.

So far I've not been able to do anything better with it. But it still seems a miracle.

Nor do I see what to do with general rectangular Young tableaux, nor if their 2-variable version has such an interpretation.

Best wishes,

Vaughan