Dear Joan, Here's a copy of a letter I just sent to Lou Kauffman

Best wishes,

Vaughan

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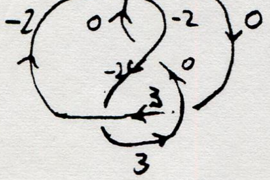
Dear Lou,

Since I'm about to talk about it today, I thought I should let you know of a states model for the two variable polynomial. It is along the lines we discussed in Geneva, there being a separate model for each of an infinite family of one variable specializations. But let me define the models first and bullshit afterwards:

<u>Notation</u>: $L$ will mean an oriented link projection of an oriented link $\mathcal{L}$, $L^*$ will denote $L$ minus the double points (crossings). The integer $n \geq 0$ is fixed.

<u>Definition</u> 1) A <u>state</u> of $L$ is a continuous function $\sigma : L^* \to \mathcal{H}_n$ where $\mathcal{H}_n$ (the set of "heights" in statistical mechanics jargon) is the set $\{-n, -n+2, -n+4, \ldots, n-2, n\}$ (eg $\mathcal{H}_2 = \{-2, 0, 2\}$). Thus a state is the assignment of any element of $\mathcal{H}_n$ to each edge of the graph underlying $L$, eg

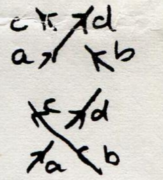



i.e. assignments to the generators in the Dehn presentation of $\pi_1(S^3-L)$

2) Given a state $\sigma$, a crossing $c$ of $L$ will be said to be of type

$$\begin{cases} 0 & \text{if } \{a,b\} \neq \{c,d\} \\ 1 & \text{if } a=d,\ b=c,\ a \neq b \\ 2 & \text{if } a=b=c=d \\ 3 & \text{if } a=c,\ b=d \quad a \neq b \end{cases}$$

, where $\sigma$ at $c$ looks like:




(positive crossing)

(negative crossing)

Let $\mathcal{C}_i(\sigma)$ be the set of crossings of $L$ of type $i$, $i = 0, 1, 2, 3$.

3) A state $\sigma$ is said to be <u>contributing</u> if

(i) $\mathcal{C}_0(\sigma) = \emptyset$

(ii) $a > b$ at positive type 3 crossings and $a < b$ at negative type 3 crossings (picture as in 2) )

<u>notes</u> If $n=1$, condition 3(i) is the ice type or six vertex condition as in [Baxter] so this is the "arrow covering" version of your model for $V$ when $n=\underline{1}$.

Condition 3(ii) is intriguing – in order to establish equivalence with your model for $V$ when $n=1$ one must allow states violating this condition but assign them weights of $0 = 1-1$. ! An easy way to remember 3(ii) is that $\sigma$ must be <u>decreasing on the overpass.</u>

4) Given a subset $X$ of the crossings of $\mathcal{L}$, $X^+$ will denote the positive ones and $X^-$ the negative ones.

$$w(X) = |X^+| - |X^-|$$

5) Assuming $L$ is $C^\infty$, let $d\theta$ denote the 1-form on $L$ giving the variation of the angle of the tangent vector to $L$.

6) If $L'$ is any oriented link, $\mathrm{rot}(L')$ will be the rotation number ($= \int_{L'} d\theta$)

7) $L^\sigma$ is $L$ with type 3 crossings smoothed.

<u>Definition</u>

$$P_L(n,s) = \frac{(-1)^{\mathrm{rot}(L)-1}\, s^{(n+1)w(L)}}{\left(\sum_{k\in\mathcal{H}_n} s^k\right)} \sum_{\sigma\in\{\text{contributing states}\}} (-1)^{|\mathcal{E}_3(\sigma)^-| + |\mathcal{E}_2(\sigma)|}\, s^{-w(\mathcal{E}_2(\sigma)) - \int_{L^\sigma} \sigma\, d\theta}\, (s-s^{-1})^{|\mathcal{E}_3(\sigma)|}$$

<u>Notes</u> a) the signs are still not ideal but I prefer to leave this formula for the moment rather than fiddle & risk screwing it up

b) Practical evaluation of $\int_{L^\sigma} \sigma\, d\theta$ is as follows: smooth $L$ at all the type 3 crossings ( → ). Then on the link projection $L^\sigma$ so obtained, $\sigma$ is continuous, i.e. constant on components, and $\int_{L^\sigma} \sigma\, d\theta = \int_{L^\sigma} \sigma\, d\theta = \sum_{\substack{\text{components } K\\ \text{of } L^\sigma}} \sigma(K)\,\mathrm{rot}(K)$.

And $\mathrm{rot}\,K$ itself is conveniently (at least for me) evaluated by smoothing all crossings and adding up the Seifert circles counted according to their orientation $\circlearrowleft = +$, $\circlearrowright = -$.

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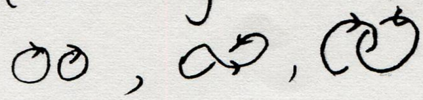
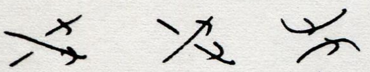
Then one may show (not entirely without pain - I suggest a few warm up exercises -  first) that $P_L(n,s)$ is invariant under the Reidemeister moves and so defines a link invariant $P_L(n,s)$. Much easier is the skein relation: if $L_+$, $L_-$ and $L_0$ are identical except in one crossing where they are as usual:

 , then $s^{-(n+1)} P_{L_+}(n,s) - s^{n+1} P_{L_-}(n,s) = (s - \frac{1}{s}) P_{L_0}(n,s)$

$L_+$ $L_-$ $L_0$

So $P(0,s)$ is trivial and $P(1,s^2) = V(s^2)$. This of course establishes the existence of the two variable polynomial in a very direct way.

You see that this is not a states model for the 2 variable poly in the sense you were trying to construct - indeed it indicates that such a model is unlikely to exist - more and more ad hoc rules will have to be added as more and more of the $P(n, \,)$'s are required to determine the 2 variable polynomial (we know from Murakami/Yamada that the smallest number of these is bounded above by the number of Seifert circles). The only hope I see for a model as you were trying to construct is if the model I have outlined can be summed in a very clever way so that the dependence on $n$ is not so explicit, say as in the expression of the Potts model partition function as a dichromatic polynomial.

The model is very suggestive as to the "real" meaning of the polynomials: $L$ should be replaced by a link in 3-space, the "states" by functions from $L$ to an $(n+1)$ dimensional Hilbert space (perhaps distributions are necessary) and the sum over contributing states by an integral with respect to some Wiener measure of an interaction term depending on the Link in $\mathbb{R}^3$. Thus it is an object of <u>gauge quantum field theory on L</u>, the gauge group in this case being $SU(n+1)$. I am morally sure that if one replaces the gauge group by $SO(2n+1)$ one will obtain the Kauffman polynomial. And there should be other poly's for all the Coxeter Dynkin diagrams, $P_L(n, s)$ being the $A_n$ series. Thus the 2 variable poly is a kind of generating function for the $A_n$ series and the Kauffman is the same for the $B_n$ series. The ubiquity of $V_L$ is explained by the fact that it corresponds ~~to~~ $SU(2)$ which is of course everywhere dense in this stuff. (eg. $A_1 = B_1 = C_1 = D_1$ if one wants to define $B_1, C_1, D_1$)

This may seem like wishful thinking based only on the states model but I haven't yet explained how I got it so let me do that and you'll see why I'm so sure of the above ideas. One must go back to the braid - Hecke - von Neumann point of view. Recall that the only values of $\tau$ for which the Hecke algebra with presentation

$$e_i^2 = e_i\ ,\ e_i e_{i+1} e_i - \tau e_i = e_{i+1} e_i e_{i+1} - \tau e_{i+1}\ ,\ e_i e_j = e_j e_i\ |i-j| \geq 2\ ,\ i = 1, 2, \ldots n$$

admits a quotient with a $C^*$ structure for all $n$ are $\tau^{-1} \geq 4$ or $\tau^{-1} = 4\cos^2 \pi/k$ and the only values of $(\tau, \eta)$ for which the Markov trace defined by $tr(x e_{n+1}) = \eta\, tr(x)$ where $x \in alg(1, e_1, \ldots, e_n)$ is positive definite on the $C^*$ algebra, are given by the Ocneanu spectrum:

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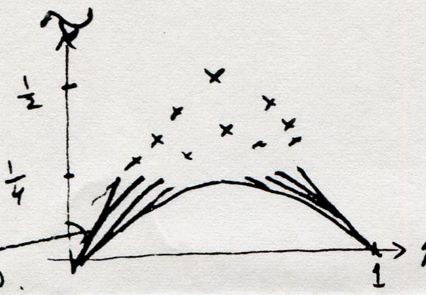


the exceptional lines.

Of particular interest here are the "exceptional lines" defined by equations which you can find in my "Notices" article. In terms of the 2-variable polynomial they corresponds to the specializations I have already described. The point is that there are particularly simple representations of the Hecke algebra (thus of the braid group) which exhibit the von Neumann algebra structure. The case $n=1$ is the Pimsner Popa Temperley Lieb representation. The higher ones were written down for me by Hans Wenzl but occurred earlier in works by Jimbo, Drinfeld & people in the Fadeev group doing quantum inverse scattering - see Drinfeld's congress talk. From this point of view they arise from "quantization" of Lie groups - this case being $SU(n+1)$. They have also written down the formulae for $B_n, C_n, D_n$ series. The only obstruction to defining the states model for the Kauffman polynomial is the precise understanding of the meaning of the Yang-Baxter (= Star triangle) equation from the braid point of view. This would also clear up the relation between Birman-Wenzl's work and the Brauer algebra.

Anyway, to come back down to earth, the representations are given ~~by~~: $e \in M_{n+1}(\mathbb{C}) \otimes M_{n+1}(\mathbb{C})$, $e = \frac{1}{1+q}\left(\sum_{i<j} e_{ii}\otimes e_{jj}\right) + \frac{q}{1+q}\sum_{i>j} e_{ii}\otimes e_{jj} + \frac{\sqrt{q}}{1+q}\sum_{i\neq j} e_{ij}\otimes e_{ji}$ where $e_{ij}$ are matrix units Then in $M_{n+1}(\mathbb{C})\otimes M_{n+1}(\mathbb{C})\otimes M_{n+1}(\mathbb{C})\otimes\ldots$

define $e_i = 1 \otimes 1 \otimes \cdots \otimes \underbrace{e}_{\substack{i\text{th}, i+1\text{th} \\ \text{slots}}} \otimes 1 \otimes 1 \cdots$ $\quad \left(s^2 = q,\ \tau = \frac{q}{(1+q)^2}\right)$

Moreover for the trace let $h = \frac{s - s^{-1}}{s^{n+1} - s^{-(n+1)}} \begin{pmatrix} s^{-n} & & 0 \\ & s^{-n+2} & \\ 0 & \ddots & s^n \end{pmatrix}$ and $H = \bigotimes_{i=1}^{\infty} (h)_i$

$tr(x) = \text{trace}(Hx)$ for $x \in \text{alg}(e_i\text{'s})$. Then if $P_L(n,s)$ is defined by the skein relation I have known for a long time that

$P_L(n,s) = (\text{normalisation})\, tr(\alpha)$, where $\alpha$ is a braid representative ($\hat{\alpha} = L$)

the model came from writing out an explicit expression for $tr(\alpha)$ by the weighted sum of the diagonal elements. The tricky part was to get that $\int_{L^*} \sigma \, d\theta$ which corresponds precisely to the weight of a diagonal element.

Precisely the same path will lead to a states model for the Kauffman polynomial, using different solutions of the Yang-Baxter equation.

One last word - the relation with the fundamental group seems rather suggestive but puzzling at this stage. Converting the "vertex model" described above to an "IRF" model on the planar dual, we see that the states assign numbers to the generators in the Dehn presentation of the fundamental group. This suggests a relationship I have long suspected between $V$ and reps of $\pi_1(S^3 - L)$ into $SU(2)$ - tying up hopefully with Casson's invariant

Best wishes,

Vaughan.