Hypothesis: $\mathcal{D}$ is a countable null decomposition of an n-sphere, X, with projection map $p: X \longrightarrow Z$.

Z is also an n-sphere, let $d \subset Z$ be the image of the nondegenerate elements of $\mathcal{D}$. Assume that the closure $\bar{d} \subset Z$ is nowhere dense.

Theorem $\mathcal{D}$ is shrinkable

pf we will approximate p by a homeomorphism p_{limit}. The approximation aspect will be supressed in our construction of p_{limit}; at the end one must check that the whole construction of p_{limit} can be done arbitrarily near p.

We need a general position lemma

Lemma let (M, ∂) be a top manifold with ∂ and C_1, C_2 countable sets in interior M. Let N_i be a nowhere dense subset of M containing C_i, $C_i \subset N_i \subset M$, for $i = 1$ and 2

Then $\forall \epsilon$ there is a homeo $h: M \longrightarrow M$ which is id on ∂M and $\sup \operatorname{dist}(h, id) < \epsilon$ and $h(C_1) \cap N_2 = \phi = h(N_1) \cap C_2$.

proof using the metric $\sup(h) + \sup(h^{-1})$. $\mathrm{Homeo}(M;\partial)$ is complete. The desired conditions are countable so the Bare Category thm applies.

<u>proof of Theorem:</u>

Begin with the following ~~commutative~~ diagram of maps between three n-spheres

$$\begin{tikzcd} X \arrow[rr, "p_0"] \arrow[dr, "q_0"'] & & Z \arrow[dl, "r_0"] \\ & W & \end{tikzcd}$$

To start with $p = p_0 = q_0$ and r_0 is "identity".

As the argument proceeds the subcripts will increase (together) though the positive integers. For all i q_i, r_i will be cell-like maps (honest functions).

For each i there will be a decompositon $\mathcal{D}_i$ of X and $\mathcal{E}_i$ of Z. $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{E}_0$ is trivial.

Let $*$ denote the union of nondegenerate elements of a decomposition. We will have: $\mathcal{D}_0^* \subset \mathcal{D}_1^* \subset \mathcal{D}_2^* \subset \cdots$ and $\emptyset = \mathcal{E}_0^* \subset \mathcal{E}_1^* \subset \mathcal{E}_2^* \subset \cdots$. The above diagram will have a commutivity condition: $r_i \circ p_i = q_i$ and a rather strong "matching condition" described below.

For each $i \geq 0$ There is a ~~closed codimension zero~~ open submanifold Θ_i of W. $W = \Theta_0 \supset \Theta_1 \supset \Theta_2 \supset \cdots$.

<u>The matching condition</u> is: (1)

$p_i \big|_{q_i^{-1}(W - \Theta_i)}$ is a homeomorphism onto its image (i.e. an imbedding

and (2) set $d_i = q_i(\mathcal{D}_i^*) \cap \Theta_i$ and $e_i = r_i(\mathcal{E}_i^*) \cap \Theta$

we require $d_i \cap \overline{e}_i = \emptyset = \overline{d}_i \cap e_i$.

Conditions (1) & (2) are vacuous for $i = 0$.

Rather than formulate the general inductive step I will show how to get from $i=0$ to $i=1$ and from

$i=1$ to $i=2$. The later is essentially equal to the general case.

Pick $\epsilon > 0$.

Suppose $\mathcal{D}_0$ contains m sets $\alpha_1 \cdots \alpha_m$ of diameter greater than ϵ. We will modify the diagram m times to get p_1, q_1, r_1, etc... What follows is the first step, the construction of $p_{1/m}, q_{1/m}$, etc...

Set $q_0(\alpha_1) = d \subset d_0$. Let B be a small round ball in W centered at d with $\partial B \cap d_0 = \emptyset$

~~The intermediate codimension 0 submanifold, call it~~ ~~will simply be W - int B~~. Call $q_0^{-1}(B) = L$ and $r_0^{-1}(B) = M$. L may fail to be a ball M is a nice flat ball. We will redefine $p_0|_L : L \longrightarrow M$ to get our $p_{1/m}$.

Let Y be a little round ball in W disjoint from $\overline{q(\mathcal{D}_0^*)}$ (use nowhere density) and set $q^{-1}(Y) = Y' \subset$

Define a stretching homeomorphism $s: B \longrightarrow \overline{W-Y}$

s should satisfy: 1 There is a smaller concentric round ball $\bar{U}$, $d \subset U \subset B$ and $s|_{U} = id_{U}$. 2 $\partial U \cap d_0 = \phi$. 3 $(s^{-1}(d_0) \cap \overset{out}{\overline{B-U}}) \cap \bar{d}_0 = \phi$

and 4 $s^{-1}(\bar{d}_0) \cap \overset{out}{\overline{B-U}} \cap d_0 = \phi$. (Conditions 3 & 4 are obtained using lemma 1)

Let $i: \overline{X-Y'} \rightarrow M$ be an homeomorphism extending $i|_{\partial}: \partial Y' \rightarrow \partial M$ where $i|_{\partial} = p \circ q^{-1} \circ s^{-1} \circ q$ by definition. Now define $p_{1/n}|_{L} = i \circ q^{-1} \circ s \circ q$ on $\overline{X - q^{-1}U}$ L ?

On ∂L we have $p_{1/n}|_{\partial L} = \underbrace{p \circ q^{-1} \circ s^{-1} \circ q}\circ \underbrace{q^{-1} \circ s \circ q} = p$

On $q^{-1}U$ we have $p_{1/n}|_{q^{-1}U} = i \circ q^{-1} \circ id \circ q = i \circ q^{-1} \circ q$.

Since q^{-1} is a relation $q^{-1} \circ q$ do not cancell but we "force them to" by defining $p_{1/n}|_{q^{-1}U} = i$.

If we define "intermediate" ~~codimension a~~ open submanifold

W-U

$\theta_{1/m}$ should be set equal to $X - q^{-1}U$. The first matching condition is clearly satisfied.

redefine r_0 to get $r_{1/m} = s^{-1} \circ q \circ i^{-1}$ (on M)

$= r_0$ (on $Z-M$)

(check on ∂M $r_{1/m} = s^{-1} \circ q \circ q^{-1} \circ s \circ q^{*} \circ p^{-1} = q \circ p^{-1} = r$ ~~since ... are~~)

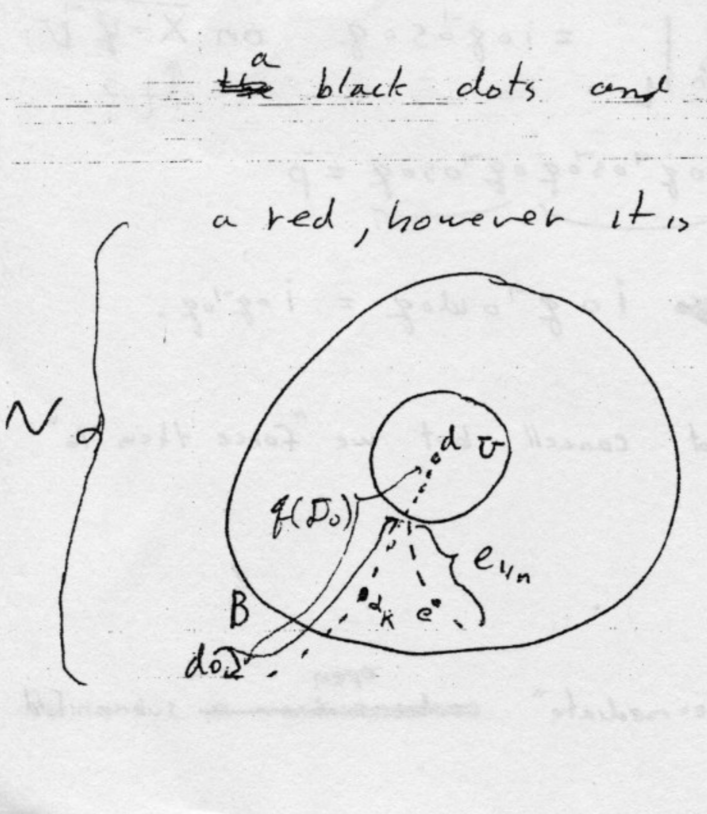
$\mathcal{E}_{1/m}$ is the decomposition determined by $r_{1/m}$.

Now consider the second matching condition.

$$e_{1/n} = s^{-1} \circ q \circ i^{-1}(i(\mathcal{D}_0^*)) = s^{-1} \circ q(\mathcal{D}_0^*) \cap \theta_{1/n} = s^{-1}(d_0) \cap B-U,$$

$(d_{1/n} = d_0 \cap B-U)$

but by ③ above $e_{1/n} \cap \bar{d}_{1/n} = \phi$. The other part of the second matching condition is similar. (In drawing pictures think of d_i as black dots in W and e_i $= s^{-1}(d_{i-1/m})$ as red dots. The red dots may not cluster at ~~the~~ a black dots and the black dots may not cluster at a red, however it is permitted for their closures to intersect.)



We are now at an intermediate stage of the first step; we have

$$\mathcal{D}_{1/m} = \mathcal{D}_0 \qquad \begin{tikzcd} X \arrow[rr, "p_{1/m}"] \arrow[dr, "q_0 = q_{1/m}"'] & & Z \arrow[dl, "r_{1/m}"] \\ \Theta_{1/m} \arrow[r, hook] & W & \end{tikzcd} \qquad \mathcal{E}_{1/m}$$

satisfying all the properties ~~stan~~ required of diagrams with integer subscripts.

If $q_0(\alpha_2), \dots, q_0(\alpha_m) \subset U$ then we are finished (that is replace subscripts $1/m$ with 1). If some $\overset{\overset{d'}{\shortparallel}}{q_0}(\alpha_k) \not\subset U$ then the previous modification must be repeated.

A new ~~Bd~~ pair of balls ~~B~~ $B' \supset U' \supset d$ is choosen so that $B' \cap \bar{E}_{1/m} = \emptyset$. This ensures that $r_{1/m}^{-1}(U')$ and $r_{1/m}^{-1}(B')$ are flat balls. The stretch map $s' : B' \to \overline{W - Y}$ is similarly defined, (construct i') set $\Theta_{2/m} = W - (U \cup U')$. Redefine p and r by the same formulas with ~~altered~~ shifted subscripts. The new collection of "black points" $d_{2/m} = d_0 - (U \cup U')$; The new collection of red points, $e_{2/m}$, is ~~[illegible]~~ $s(q\mathcal{D}_0) \cup s'(q\mathcal{D}_0) - (U - U')$.

— The new balls (B', U') should not intersect ∂B or ∂U and $\partial B', \partial B'$ should be disjoint from $d_0 = q\mathcal{D}_0$.

Finite itteration results in $p_1, q_1 = q_0, r_1$ $\mathcal{D}_1, \mathcal{E}_1$, etc....

The passage from 1 to 2 is designed to remove all sets of diameter $> \epsilon/2$ which p_1^{-1} sends to a point. The stages in this step are comparable to all but the first stage of step 1. Here we draw (red) a balls ~~about certain~~ pair (B, U) around an element $e \in e_1$ with $r_1^{-1}(e)$ having large diameter. It is important for the definition of a new ~~map~~ homeomorphism "$i : \Sigma - Y$" $\rightarrow q^{-1}(B)$ that the red Ball B be disjoint from the closure of the black points $q(\mathcal{D}_1^*)$. This ensures that $p_1^{-1}(B)$ is a (flat) ball.

Using the general position lemma it is

possible to propigate the second matching condition where

Θ will always be $W - (\bigcup v\text{'s})$.

Continuing in this way construct $\rho_1, \rho_2, \rho_3, \ldots$

ρ_{i+1} has no pt. inverses of diam $\geq \epsilon(1/2)^n$, ρ_{2n}^{-1} has no pt. inverses of diam $\geq \epsilon(1/2)^n$.

let ρ_{limit} be the limit. To prove existance as a function, continuity, and

1-1ness it is a good idea to go back and w.l.o.g.

impose a little extra controle on the red and black balls

$\{B's\} \subset W$. ① The boundaries of no two distinct

B's intersect and ② no B lies in a v (this is automatic

from construction. ③ Let us index a B by i, B_i, if it

is placed down while transforming ρ_i to ρ_{i+1} (or ρ_i^{-1} to ρ_{i+1}^{-1})

we can arrange that measured in fixed metrics on

X and Z ~~$\operatorname{diam}(f^{-1}(B_i))$, $\operatorname{diam}(r^{-1}(B_i)) < \epsilon(1/2)^i$~~

~~for $i \geq 2$~~ $\operatorname{diam} f^{-1}(B_i) < \epsilon(1/2)^i$ for $i \geq 2$ and

$\operatorname{diam} r^{-1}(B_i) < \epsilon(1/2)^i$ for $i \geq 1$.

Assume $\rho_{limit}(x_1) = \rho_{limit}(x_2)$.

If x_1 and x_2 both lie in only finitely many balls

then $\exists\ i$ larger than the subscript of any ball containing x_1 or x_2. Now ~~s.t.~~ $\rho_i(x_1) = \rho_{limit}(x_1)$ and $\rho_i(x_2) = \rho_{limit}(x_2)$.

But This implies x_1 and x_2 lie in some non trivial

point inverse of ρ_i which implies that the will be incased

in som $q^{-1}(B_j)$ for $j \geq i$. contradiction So ρ_{limit} is a

homeo morphism.

~~Since S^4 is compact ρ_{limit} is a covering~~ map.

~~Since $(n>1)$ $\pi_1(S^4) = \{e\}$ ρ_{limit} is a homeomorphism.~~

Note: This leads to topological 2-handles inside

Casson Handles.