

A GEOMETRIC FORM OF CASSON'S INVARIANT AND ITS CONNECTION TO REIDEMEISTER TORSION

1

CONTENTS by Dennis Johnson

1. GENERAL IDEA
2. CROSSED HOMOMORPHISMS AND \mathcal{C}^*
3. LOCAL HOMOLOGY FOR SURFACES AND 3-MANIFOLDS
4. LOW DIMENSIONAL COHOMOLOGY OF GROUPS
5. REPRESENTATION SPACES $\text{Hom}(\pi_1(S), G)$, $G = \text{LIE GROUP}$
6. CONJUGACY CLASS SPACES
7. MILNOR'S VOLUME FORMALISM
8. THE TANGENTIAL COMPLEX
9. THE VOLUMES ON $M^*(\mathbb{Z}^2, \pi)$ AND $Z^1(\pi, M)$ FOR π FREE
10. THE VOLUMES ON $\mathbb{R}, \hat{\mathbb{R}}$ FOR SURFACE & FREE GROUPS
11. THE VOLUME CHAIN COMPLEX OF THE TANGENTIAL COMPLEX WITH $C^*(W^3, L)$
12. THE TORSION FOR $SL(2, \mathbb{C})$ LEFS OF 2-BRIDGE KNOTS
13. THE TREFOIL TORSION POLYNOMIALS AND RECURSION

NOTE: EVERYTHING IN 2-6 IS BASICALLY STANDARD ABSTRACT NONSENSE EXCEPT THE CONSTRUCTION (& LATER USES) OF \mathcal{C}^* IN 2) AND THE SPECIFIC LOCAL CHAIN COMPLEXES FOR SURFACES AND 3-MANIFOLDS IN 3) — AND THESE ARE NEAR STANDARD. IF YOU ARE FAMILIAR WITH THE BASIC HOMOLOGICAL ALGEBRA YOU CAN SKIP MOST OF IT. THERE IS ALSO NOTHING NEW IN 7 + 9 IF YOU ARE FAMILIAR WITH REIDEMEISTER AND/OR WHITEHEAD TORSION. THE ONLY IDEAS ARE IN 1, 8 AND 10-13, WITH 8 + 10 CONSTRUCTING THE "TANGENTIAL" COMPLEX WHICH GIVES A CASSON TYPE INVARIANT, AND 11, SHOWING IT IS THE SAME AS A CERTAIN LOCAL CHAIN COMPLEX FOR THE 3-MANIFOLD. 12 & 13 DO SOME CALCULATIONS FOR SOME EASY KNOTS, FINDING THE TORSION POLYNOMIALS FOR ALL TREFOIL SURGERIES IN PARTICULAR.

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A GEOMETRIC FORM OF CASSON'S INVARIANT AND REIDEMEISTER TORION

I. THE GENERAL IDEA

LET W^3 BE AN ORIENTED 3-MANIFOLD GIVEN AS A HEEGAARD DECOMPOSITION INTO GENUS g HANDLEBODIES A, B WITH $A \cap B =$ GENUS g SURFACE K . WE HAVE THE ~~SETS~~ ^{(FINITE) MANIFOLDS} OF CONJUGACY CLASSES OF REPRESENTATIONS OF $\pi_1(A), \pi_1(B), \pi_1(K)$ INTO A LIE GROUP G , OF RESPECTIVE DIMENSIONS $(g-1)d, (g-1)d, 2(g-1)d$ WHERE $d = \dim G$, ~~AND~~ WHICH WE CALL $\hat{R}_A, \hat{R}_B, \hat{R}_K$ RESPECTIVELY. THE FIRST TWO HAVE A NATURAL IMBEDDING INTO \hat{R}_K AND WE IDENTIFY THEM WITH THESE IMBEDDINGS. FOR $G = SU(2)$, CASSON HAS DEFINED AN INVARIANT OF W^3 WHICH IS BASICALLY THE ALGEBRAIC INTERSECTION $\#$ OF $\hat{R}_A \# \hat{R}_B$ IN \hat{R}_K : THE SUBMANIFOLDS $\hat{R}_A \# \hat{R}_B$ CAN BE COHERENTLY ORIENTED OUT INTO TRANSVERSE POSITION AND THE SIGN OF THE INTERSECTIONS DETERMINED USING A NATURAL ORIENTATION ON \hat{R}_K ; THE SUM OF THESE SIGNS IS AN INVARIANT.

NOW IT TURNS OUT THAT \hat{R}_K NOT ONLY HAS A NATURAL ORIENTATION, BUT A NATURAL VOLUME AS WELL, ~~WHICH~~ I.E., A TOP ORDER FORM, WHICH WE DENOTE BY θ_K . FURTHERMORE, $\hat{R}_A \# \hat{R}_B$ ALSO HAVE NATURAL VOLUMES, DETERMINED ONLY UP TO SIGN HOWEVER; BUT JUST AS CASSON COHERENTLY ORIENTS \hat{R}_A, \hat{R}_B , WE CAN CHOOSE THE SIGNS $\pm \theta_A, \pm \theta_B$ COHERENTLY, I.E. A CHOICE OF θ_A DETERMINES A NATURAL CHOICE OF θ_B , AND CHANGING θ_A ALSO CHANGES θ_B . ~~THIS~~ THUS $\theta_A \wedge \theta_B$, WHICH IS A TOP FORM, IS INDEPENDENT OF CHOICE. SINCE θ_K ~~IS~~ IS NOWHERE ZERO, AT ANY INTERSECTION POINT p OF $\hat{R}_A \cap \hat{R}_B$ WE HAVE A WELL DEFINED NUMBER $t_p \ni \theta_A \wedge \theta_B = t_p \theta_K$. ALSO, SINCE θ_A, θ_B ARE NOWHERE ZERO, $t_p \neq 0$ IFF p IS A TRANSVERSE INTERSECTION OF $\hat{R}_A \# \hat{R}_B$. WE CALL THIS NUMBER THE TORSION OF p .

IF $G = SU(2)$ AND THE POINT p IS A TRANSVERSE INTERSECTION, WE ARE LOOKING AT AN INTERSECTION OF THE TYPE CASSON CONSIDERS. IN THIS CASE, t_p IS REAL AND NON ZERO, AND CASSON'S INVARIANT

SIGN IS JUST $\text{SIGN}(t_p)$. IN THE CASE WHEN ALL THE INTERSECTIONS ARE TRANSVERSE, CASSON'S INVARIANT IS JUST $3 \sum \text{SIGN}(t_p)$. THIS EXPRESSION MOTIVATES THE CONSIDERATION OF $\sum t_p$ ITSELF, AS A KIND OF PARALLEL FORM OF CASSON'S INVARIANT. (NOTE THAT WE NEED ONLY SUM OVER THE TRANSVERSE INTERSECTIONS, SINCE $t_p = 0$ AT A NON TRANSVERSE INTERSECTION) WE SHALL SEE THAT $\sum t_p$ IS ALSO AN INVARIANT OF THE 3-MANIFOLD W .

SEVERAL CONSIDERATIONS ARE IN ORDER HERE:

1) IF p IS TRANSVERSE, THEN t_p IS AN ALGEBRAIC NUMBER AND THE REPRESENTATION IS "ALGEBRAIC" I.E. IT CAN BE CHOSEN SO THAT ITS MATRICES ALL HAVE ENTRIES IN SOME FINITE EXTENSION FIELD F OF \mathbb{Q} . (WE ARE ASSUMING THAT G IS AN ALGEBRAIC GROUP OVER \mathbb{Q} HERE), AND THEN t_p WILL BE IN F ALSO.

2) HENCE THE GALOIS CONJUGATES OF p ARE ALSO DEFINED, WITH THEIR TORSION EQUAL TO THE CORRESPONDING CONJUGATES OF t_p . FOR THIS REASON, IT IS MORE APPROPRIATE TO TAKE $\sum t_p$ OVER ALL REPS INTO $SL(2, \mathbb{C})$, NOT JUST $SU(2)$ (OR IN GENERAL, ANY ALGEBRAIC GROUP OVER \mathbb{Q}). IF WE DO THIS, $\sum t_p$ IS INVARIANT UNDER THE GALOIS GROUP AND IS THUS A RATIONAL #.

3) GOING ONE STEP FURTHER, WE MAY AS WELL LOOK AT THE OTHER SYMMETRIC FUNCTIONS OF THE t_p 'S — IN OTHER WORDS, WE FORM THE POLYNOMIAL $\sigma(t)$ WHOSE ROOTS ARE ALL THE t_p 'S (OVER TRANSVERSE p 'S ONLY). THIS POLYNOMIAL HAS RATIONAL COEFFICIENTS AND IS ALSO AN INVARIANT OF W^3 . WE AIM TO SHOW THAT $\sigma(t)$ (AND/OA ITS ROOTS) IS A KIND OF REIDEMEISTER TORSION ON W , AND COMPUTE IT FOR SURFACES ON SOME KNOTS. THIS ESTABLISHES AN INTERESTING BUT QUITE VAGUE CONNECTION BETWEEN CASSON'S INVARIANTS AND REIDEMEISTER TORSION WHICH I DO NOT UNDERSTAND PARTICULARLY SINCE THE CONNECTION SEEMS TO BREAK DOWN COMPLETELY WHEN $\hat{R}_A \cap \hat{R}_B$ IS NOT DISCRETE.

II. Crossed Hom and dTT

A. Crossed Hom

4.

If π is a group and M is a π -module, a crossed hom $\varphi: \pi \rightarrow M$ is such that $\varphi(xy) = \varphi(x) + x\varphi(y)$. The set of crossed homs is an abelian group under pointwise addition which we designate (generalizing) by $X\text{HOM}(\pi, M)$. The law above implies easily that $\varphi(1) = 0$ and $\varphi(xx^{-1}) = \varphi(x) + x\varphi(x^{-1})$, i.e. that $\varphi(x^{-1}) = -x^{-1}\varphi(x)$. Also, it is clear inductively that ~~if~~ if x_i are elements of π and w is a word in the x_i 's, then we can "expand" $\varphi(w)$ as a $\mathbb{Z}\pi$ linear combination of $\varphi(x_i)$.

EXAMPLES:

1. If $\varphi: \pi \rightarrow M$ is crossed and $f: M \rightarrow N$ is a π -module map, then $f\varphi: \pi \rightarrow N$ is crossed.

2. $\varphi: \pi \rightarrow \mathbb{Z}\pi$ by $\varphi(x) = x - 1$ is crossed:

$$\varphi(xy) = xy - 1 = (x - 1) + x(y - 1)$$

3. Generalizing 2, if M is any π -module and $m \in M$, define $\varphi_m: \pi \rightarrow M$ by $\varphi(x) = (x - 1)m$. Such a crossed hom is called a principal crossed hom (example 2 is just φ_1)

4. Suppose π acts linearly on an affine space A , i.e. it acts on A and also acts linearly on A 's vector space of translations V in such a way that $g(q + v) = g(q) + g(v)$ [for example, let $A = \text{space of connections}$, $V = 1$ -forms, $\pi = \text{gauge group}$]

Define for $q \in A$, $\varphi_q: \pi \rightarrow V$ by $\varphi_q(g) = g(q) - q$.

[This is not necessarily principal, because q_0 is in A , not V .] Then set

$$\varphi_q(g) = g(q) - q = g(h) - g(q) + g(q) - q = \varphi_q(g) + g\varphi_q(h)$$

Note that if we choose a different point q_1 as "base point", we

have $\varphi_{q_1}(g) - \varphi_{q_0}(g) = (g^{-1})v$, where $v = q_1 - q_0$ is in V —

that is, $\varphi_{q_1} - \varphi_{q_0}$ is a principal crossed hom. So φ is well defined mod princip. crossed homs.

Our next example is the most important one

B. THE FOX FREE DERIVATIVES $\frac{\partial}{\partial x_i}$

Let F be a free group on (x_1, \dots, x_n) ; we investigate the crossed homomorphisms $\varphi: F \rightarrow ZF$. Let the value of φ at x_i be arbitrarily specified to be φ_i ; then there is a unique extension to all of F . Clearly it is unique since φ_i determines φ on all words. To see that it is well defined, note that two words are equal in F iff they differ by moves of the form $W_1 W_2 \leftrightarrow W_1 x_i x_i^{-1} W_2 \leftrightarrow W_1 x_i^{-1} x_i W_2$

Applying the proposed definition of φ to the 3 words gives $\varphi_1(W_1) + W_1 \varphi(W_2)$, $\varphi(W_1) + W_1 \varphi(x_i) = W_1 x_i (-x_i^{-1} \varphi(x_i) + W_1 x_i^{-1} \varphi(W_2))$ and similarly for the 3rd word; the results are all equal in ZF .

Thus φ exists and is unique. In particular, define

$$\frac{\partial}{\partial x_i}: F \rightarrow ZF \quad \text{by} \quad \frac{\partial x_j}{\partial x_i} = \delta_{ij} \quad \text{These crossed homomorphisms}$$

are Fox's free derivatives, and we see that they form a basis for $X\text{Hom}(F, ZF)$ as a ZF -module, where the action of π on $X\text{Hom}$ is given by right multiplication, i.e. $x\varphi: F \rightarrow ZF$ is given by $W \rightarrow \varphi(W) \cdot x^{-1}$

Note that if π is any group, $x_1, \dots, x_n \in \pi$ and W is a word in x_i 's, $\frac{\partial W}{\partial x_i}$ still makes sense as an element of $Z\pi$,

and we have, for a crossed homomorphism $\varphi: \pi \rightarrow M$, that

$$\varphi(W) = \sum_i \frac{\partial W}{\partial x_i} \varphi(x_i)$$

PROP. 1 Let π be generated by x_i with relations $r_j(x_i)$.

If M is a π -module with $m_i \in M$, then $\varphi(x_i) = m_i$ defines a (unique) crossed homomorphism $\pi \rightarrow M$ iff $\sum_i \frac{\partial r_j}{\partial x_i} m_i = 0$ in M for all j .

PROOF: It is clearly necessary, since $0 = \varphi(1) = \varphi(r_j(x_i)) = \sum_i \frac{\partial r_j}{\partial x_i} \varphi(x_i) = \sum_i \frac{\partial r_j}{\partial x_i} m_i$

WE NEED ONLY SEE THAT IT IS WELL DEFINED ON π .

SUPPOSE TWO WORDS $w_1(x_i)$ AND $w_2(x_i)$ ARE EQUAL IN π ;

THEN $w_1 w_2^{-1} = \prod_k u_k r_k u_k^{-1}$, WHERE r_k IS ONE OF THE r_i 'S.

WE HAVE THE FOLLOWING:

1) $\varphi(r_k) = \sum \frac{\partial r_k}{\partial x_i} \varphi(x_i) = \sum \frac{\partial r_k}{\partial x_i} x_i = 0$

2) $\varphi(u_k r_k u_k^{-1}) = \varphi(u_k) + \varphi(r_k) - \varphi(u_k) = 0$

3) $\varphi(\prod_k u_k r_k u_k^{-1}) = \varphi(u_1 r_1 u_1^{-1}) + \varphi(u_2 r_2 u_2^{-1}) + \dots = 0$

4) $\varphi(w_1) = \varphi(w_2 \prod_k u_k r_k u_k^{-1}) = \varphi(w_2) + \varphi(\prod_k u_k r_k u_k^{-1}) = \varphi(w_2)$

SO φ IS WELL DEFINED, QED

C. THE MODULE $d\pi$

AT FIRST START, THE CHOICE OF SYMBOLS $\frac{\partial}{\partial x_i}$ SEEM TO HAVE BEEN MADE ONLY BECAUSE THEY FOLLOW THE "HALF-LEIBNIZ" LAW OF CROSSED HOMOMORPHISMS, BUT THERE IS MORE RELATION TO CALCULUS THAN THAT. CONTINUING THE PARALLEL WE DEFINE A NATURAL π -MODULE ASSOCIATED TO π WHICH WILL BE USED FREQUENTLY IN THE FOLLOWING.

LET $d\pi$ BE THE π -MODULE WITH A GENERATOR dx FOR EACH $x \in \pi$ AND RELATIONS $d(xy) = dx + x dy$ FOR ALL $x, y \in \pi$. WE HAVE

THE FOLLOWING FACTS:

1. $d(1) = 0$ AND $d(x^{-1}) = -x^{-1} dx$

2. THE MAP $d: \pi \rightarrow d\pi$ GIVEN BY $x \rightarrow dx$ IS A CROSSED HOMOMORPHISM, JUST BY DEFINITION

3. IF $x_i \in \pi$ AND w IS A WORD IN THE x_i 'S, THEN $dw = \sum \frac{\partial w}{\partial x_i} dx_i$. THIS FOLLOWS FROM THE LAW $\varphi(w) = \sum \frac{\partial w}{\partial x_i} \varphi(x_i)$ BY APPLYING IT TO THE CROSSED HOM

4. GIVEN ANY CROSSED HOM $\varphi: \pi \rightarrow M$, THERE IS A UNIQUE ~~...~~ 7.
 $\mathbb{Z}\pi$ -MOD $f: d\pi \rightarrow M$ SUCH THAT $\begin{array}{ccc} \pi & \xrightarrow{\varphi} & M \\ \downarrow d & \searrow f & \\ d\pi & \xrightarrow{f} & M \end{array}$ COMMUTES — NAMELY,

WE MUST DEFINE $f(dx) = \varphi(x)$ SO WE NEED ONLY SEE THAT IT IS WELL
 DEFINED I.E. THAT $f(dx_1 dx_2 - x_1 dx_2) = 0$. BUT THE LEFT SIDE
 IS $\varphi(x_1 x_2) - x_1 \varphi(x_2) - \varphi(x_1) x_2 = 0$. THE ASSOCIATES $\varphi \rightarrow f, f \rightarrow \varphi$
 ARE THEREFORE ~~...~~ INVERSE TO EACH OTHER, AND HENCE
 WE HAVE AN INDUCED ISOMORPHISM $\text{Hom}(d\pi, M) \xrightarrow{d} \mathbb{X}\text{Hom}(\pi, M)$. (WE
 FREQUENTLY IDENTIFY THE TWO GROUPS VIA THIS COMPOSITION WITH d .)

5. THE PRINCIPAL CROSSED HOMOMORPHISM $\pi \rightarrow \mathbb{Z}\pi$ SENDS x TO $x-1$
 INDUCES THE MODULE HOM $\epsilon: d\pi \rightarrow \mathbb{Z}\pi$ GIVEN BY $\epsilon(dx) = x-1$ FOR ALL $x \in \pi$
 IF w IS A WORD IN $x_i \in \pi$, THIS GIVES THE "EULER FORMULA":

$$w-1 = \epsilon(dw) = \epsilon\left(\sum \frac{\partial w}{\partial x_i} dx_i\right) = \sum \frac{\partial w}{\partial x_i} \epsilon(dx_i) = \sum \frac{\partial w}{\partial x_i} (x_i - 1)$$

I.E.:

Prop. 3:
$$\sum \frac{\partial w}{\partial x_i} (x_i - 1) = w - 1$$

Prop. 4

6. A IF F IS A FREE GROUP ON $\{x_1, \dots, x_n\}$, $d\pi^F$ IS FREE ON $\{dx_i\}$
 IN FACT, $d\pi^F$ IS CLEARLY GENERATED BY dx_i ($d\pi = \sum \frac{\partial \pi}{\partial x_i} dx_i$)
 ALSO, THE DERIVED NEWS $\frac{\partial}{\partial x_i}: F \rightarrow \mathbb{Z}F$ INDUCE CORRESPONDING $\mathbb{Z}F$ -HOMS

$$d\pi^F \rightarrow \mathbb{Z}F, \text{ WHICH WE DENOTE BY } \frac{\partial}{\partial x_i} \text{ AND } \frac{\partial}{\partial x_i}(dx_j) = \delta_{ij}$$

SUPPOSE THEN THAT $\sum r_j dx_j$ WERE 0 IN $d\pi^F$.
 APPLYING $\frac{\partial}{\partial x_j}$, WE GET $r_j = 0$ IN $\mathbb{Z}F$, ALL j .
 I.E., THE ONLY RELATION AMONG THE dx_i 'S IS THE TRIVIAL ONE.

D. CHANGING COEFFICIENTS ON $d\pi$

LET $h: \pi \rightarrow \pi$ BE A GROUP HOMOMORPHISM. WE CAN MAKE
 A π -MODULE $\pi d\pi$ BY THE SAME METHOD AS WE MADE $d\pi$: GENERATORS dx ($x \in \pi$)
 BUT WITH RELATIONS $d(xy) = dx + h(x)dy$. THIS IS EXACTLY THE SAME

At $Z\pi \otimes d\pi$. HERE π ACTS ON THE RIGHT OF $Z\pi$ THROUGH THE...
~~homomorphism~~ k AND $X \otimes y dz = X h(y) \otimes dz$ HOLDS IN $Z\pi \otimes d\pi$.
 THE EQUALITY CONVERTS THE RELATIONS IN $d\pi$ TO THOSE IN $Z\pi \otimes d\pi$. WE

WILL MAKE FREQUENT USE OF THIS CONSTRUCTION, AND MAY ABBREVIATE
 $Z\pi \otimes d\pi$ TO ~~XXXXXXXXXX~~ $\pi \otimes d\pi$.

WE HAVE A WELL DEFINED π -MODULE HOM $dh: \pi \otimes d\pi \rightarrow d\pi$
 GIVEN BY $dh(dx) = \cancel{h(x)} d(h(x))$, SINCE THE REQUIRED RELATIONS HOLD:

$\rightarrow dh(d(xy) - dx - h(x)dy) = d(h(xy)) - d(h(x)) - h(x)d(h(y)) = 0$ IN $d\pi$

EXAMPLE: LET $h: \pi \rightarrow \pi$ BE THE TRIVIAL HOM; THEN $dh = 0$

PROOF: $dh(dx) = d(h(x)) = d(1) = 0$ FOR ALL $x \in \pi$

WE NOW HAVE "d" DEFINED ON BOTH GROUPS AND HOMOMORPHISMS.

IT IS FUNCTORIAL IN THE FOLLOWING SENSE: IF $\pi \xrightarrow{h} \pi \xrightarrow{k} P$ THEN

$ZP \otimes d\pi \xrightarrow{dP \otimes dh} ZP \otimes d\pi \xrightarrow{dP} dP$ EQUALS $ZP \otimes d\pi \xrightarrow{d(kh)} dP$

JUST CHASE THE DIAGRAMS. WE ABBREVIATE: $\pi \otimes d\pi \xrightarrow{dh} \pi \otimes d\pi \xrightarrow{d(kh)} dP$

LATER WE WILL SEE THAT $d\pi$ IS LIKE AN ABSTRACT COTANGENT SPACE OF SOMETHING, $\pi \otimes d\pi$ IS THE COTANGENT SPACE RESTRICTED TO THE PLACES WHERE SOME EXTRA RELATIONS HOLD (NAMELY, THE NEW ONES IN π) AND THE MAP $\pi \otimes d\pi \rightarrow d\pi$ IS THE RESTRICTION OF THESE DIFFERENTIALS TO THE "TANGENT SPACE" OF π ITSELF.

WARNING: LET $\pi = \pi$ AND $h: \pi \rightarrow \pi$, $dh: d\pi \rightarrow d\pi$; dh IS NOT

A MODULE MAP HERE, BECAUSE IT IS "CHANGING THE COEFFICIENTS" — I.E., WE HAVE $dh(xdy) = h(x) d(h(y))$, NOT $= x d(h(y))$ AS IT SHOULD BE.

G.I.F.: $dh(xdy) = d(h(x)) + h(x)d(h(y))$

EXAMPLES: LET F BE FREE ON X AND LET A BE ITS ABELIANIZATION, WITH CORRESPONDING GENERATORS a, b . SUPPOSE $f: F \rightarrow F$ IS A HOM WHICH INDUCES 1 ON H (I.E., "TORELLI GROUP"). THEN

~~XXXXXXXXXX~~ $df: \pi \otimes dF \rightarrow \pi \otimes dF$ IS A MODULE HOM:

.. $df(a dx) = f(a) d(fx) = a d(fx) = a df(x)$ AS REQUIRED.

9.

For example, consider f given by $f(x) = x$, $f(y) = y^2 x y^{-1} x^{-1}$.
 WE GET $f(a) = a$, $f(b) = b$ so $f = 1$ ON A ; AND

$dx \rightarrow dx$

$dy \rightarrow dy + b dy + b^2 dx - ab dy - b dx$
 $= (b^2 - b) dx + (1 + b - ab) dy$

i.e. $\begin{pmatrix} dx \\ dy \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ b^2 - b & 1 + b - ab \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$. SINCE THE DETERMINANT IS $1 + b - ab$,

WHICH IS NOT A UNIT IN $\mathbb{Z}A$, f IS NOT INVERTIBLE.

E. PRESENTATION OF $d\pi$

SUPPOSE $R \xrightarrow{f} F \xrightarrow{p} \pi \rightarrow 0$ IS A PRESENTATION OF π , I.E.

F IS FREE ON $\{x_i\}$

R " " " $\{y_j\}$

p IS ONTO

$y_j = g(y_j)$ NORMALLY GENERATE $\text{Ker } p$

THEN:

PROB. 5 $\mathbb{Z}\pi \otimes R \xrightarrow{\mathbb{Z}\pi \otimes f} \mathbb{Z}\pi \otimes dF \xrightarrow{dp} d\pi \rightarrow 0$ IS EXACT,

WHERE $\mathbb{Z}\pi$ IS A MODULE PRESENTATION OF $d\pi$.

PROOF: FACT, $dp_*(\mathbb{Z}\pi \otimes g) = d(p \circ g) = d(\text{TRIV}) = 0$;

HENCE IF WE WRITE $M = \text{Coker}(\mathbb{Z}\pi \otimes g)$, dp INDUCES $\mu: M \rightarrow d\pi$.

TO SAY THE SEQUENCE IS EXACT IS JUST TO SAY THAT μ IS AN ISOMORPHISM.

TO SEE THIS, LET $p(x_i) = r_i \in \pi$, SO THAT $\{r_i \mid r_j(r_j)\}$ IS A PRESENTATION

OF π . ALSO, LET THE IMAGE OF $dx_i \in dF$ IN M

BE CALLED m_i , SO THAT $\mu(m_i) = dr_i$.

THEN WE DEFINE A MODULE MAP $f: \mathcal{D} \rightarrow M$ BY $f(dx_i) = x_i$: IF THIS DEFINITION GOES THROUGH, f AND f^{-1} ARE INVERSES AND WE ARE DONE.

10.

TO SEE THAT THE DEFINITION IS OK WE USE PROPOSITION 1: WE NEED ONLY VERIFY THAT $\sum \frac{\partial r_j}{\partial x_i} m_j = 0$ IN M . THIS IS THE IMAGE OF

$$\frac{\partial y_j(x_i)}{\partial x_i} dx_i \text{ IN } \mathbb{Z}^n \otimes \mathcal{D}, \text{ i.e. OF } dr_j; \text{ BUT } dr_j \in \ker d\phi,$$

SO IT GOES TO 0 IN M , Q.E.D.

F. TENSOR PRODUCT AND RIGHT π -MODULES.

EXCEPT FOR AN OCCASIONAL \mathbb{Z}^n , ALL OUR MODULES TO THIS POINT HAVE BEEN LEFT π -MODULES, I.E. THEY SATISFY $(xy)m = x(y m)$:

BEFORE DOING LOCAL HOMOLOGY WITH COEFFICIENTS WE NEED TO TALK ABOUT THE TENSOR PRODUCT OF TWO π -MODULES OVER \mathbb{Z}^n . LET M BE A LEFT π -MODULE AS USUAL, AND P BE A RIGHT π -MODULE — THAT IS, π ACTS ON THE RIGHT OF P : $p x$ IS DEFINED FOR $p \in P, x \in \pi$, SATISFYING NOW THE ASSOCIATIVE LAW $p(xy) = (px)y$.

EXAMPLES: 1) IF M IS LEFT, WE GET AN ASSOCIATED RIGHT MODULE \bar{M} BY DEFINING $\bar{M} = M$ AS A SET BUT DEFINING $\pi x = x^{-1} m$. NOTE THAT π IS ACTING ENTIRELY DIFFERENTLY ON M , AND IN FACT \bar{M} CANNOT (USUALLY) BE VIEWED AS A LEFT π -MODULE IN ANY NATURAL WAY.

2) IF M IS A VECTOR SPACE, THEN ITS DUAL VECTOR SPACE M^* HAS A NATURAL RIGHT MODULE STRUCTURE BY DEFINING πx TO BE THE UNIQUE THING SATISFYING $\langle \pi x, m \rangle = \langle m^*, x m \rangle$ FOR ALL $m \in M$.

3) IF M HAS A π -INVARIANT BILINEAR FORM \langle, \rangle WHICH IS NONSINGULAR, THEN M ACQUIRES A RIGHT STRUCTURE BY EITHER PROCESS 1) OR PROCESS 2) V.I.A IDENTIFYING M + M^* . BUT THESE ARE ACTUALLY EQUAL, SINCE BY DEFINITION 2), $\langle \pi x, m \rangle = \langle m, x m \rangle = \langle x^{-1} m, m \rangle$ I.E. $\pi x = x^{-1} m$, WHICH IS THE SAME AS DEFINITION 1).

WARNING: THE COMBINED LEFT AND RIGHT STRUCTURES ON M IS NOT 11.

IN GENERAL ASSOCIATIVE, I.E. $x(my) = x y^{-1} m$, BUT $(xm)y = y^{-1} x m$.
 (IN FACT, IF M HAS L & R STRUCTURES WHICH ASSOCIATE IT IS CALLED A BIMODULE):

RETURNING TO OUR MODULES $M \triangleleft P$ WE DEFINE THE TENSOR PRODUCT $P \otimes_{\pi} M$ OVER $\mathbb{Z}\pi$ TO BE $P \otimes M$ AS A GROUP BUT WITH THE ADDED RELATIONS $p \otimes x m = p x \otimes m$ FOR $x \in \pi$.

→ RECALL THAT IF $A \rightarrow B \rightarrow C \rightarrow 0$ IS AN EXACT SEQUENCE OF LEFT π -MODULES, AND P, M ARE LEFT, RIGHT π -MODULES, THEN:

1) $P \otimes_{\pi} A \rightarrow P \otimes_{\pi} B \rightarrow P \otimes_{\pi} C \rightarrow 0$ IS EXACT.

2) $\text{Hom}_{\pi}(A, M) \leftarrow \text{Hom}_{\pi}(B, M) \leftarrow \text{Hom}_{\pi}(C, M) \leftarrow 0$ IS EXACT

APPLYING THESE TO PROPOSITION 5, WE GET

PROP 6

$P \otimes_{\pi} dR \xrightarrow{d\pi} P \otimes_{\pi} dF \xrightarrow{d\pi} P \otimes_{\pi} d\pi \rightarrow 0$ IS EXACT, GIVING A REPRESENTATION OF \mathbb{R}

AND $\text{Hom}_{\pi}(dR, M) \xleftarrow{d\pi^*} \text{Hom}_{\pi}(dF, M) \xleftarrow{d\pi^*} \text{Hom}_{\pi}(d\pi, M) \leftarrow 0$ IS EXACT

I.E., $X\text{Hom}_{\pi}(R, M) \xleftarrow{d\pi^*} X\text{Hom}_{\pi}(F, M) \xleftarrow{d\pi^*} X\text{Hom}_{\pi}(\pi, M) \leftarrow 0$ IS EXACT

[NOTE: HERE, $P \otimes_{\pi} dR$ REALLY MEANS $P \otimes_{\pi} (\mathbb{Z}\pi \otimes dR)$, AND LIKEWISE $\text{Hom}_{\pi}(dR, M)$ MEANS $\text{Hom}_{\pi}(\mathbb{Z}\pi \otimes dR, M)$

G. DUALITY

LET M BE A VECTOR SPACE OVER F ON WHICH π ACTS ON THE LEFT. THEN M^* IS A RIGHT π -MODULE AS WE SAW ABOVE. THERE IS AN F -BILINEAR PAIRING OF $M^* \otimes_{\pi} M$ WITH $\text{Hom}_{\pi}(d\pi, M) \cong X\text{Hom}_{\pi}(\pi, M)$ GIVEN BY:

$\langle m^* \otimes dx, f \rangle = \langle m^*, f(dx) \rangle$ FOR $f \in \text{Hom}_{\pi}(d\pi, M)$

OR EQUIVALENTLY, $\langle m^* \otimes dx, \varphi \rangle = \langle m^*, \varphi(x) \rangle$ FOR $\varphi \in X\text{Hom}_{\pi}(\pi, M)$.

TO BE SURE THIS DEFINITION IS OK, WE MUST GET THE SAME RESULT

ON $M^* \otimes dy$ AND $M^* \otimes x dy$, SINCE THESE ARE EQUAL IN $M^* \otimes d\pi$.
 \rightarrow BUT THE FIRST GIVES $\langle M^* \otimes dy, f \rangle = \langle M^* x, f dy \rangle = \langle M^*, x f dy \rangle$
 AND THE SECOND $\langle M^*, f x dy \rangle = \langle M^*, x f dy \rangle$. 12.

PROP 7 THE ABOVE PAIRING IS DUAL, I.E. NONSINGULAR.

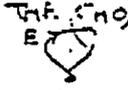
PROOF: TO SEE THAT THE MAP $\text{Hom}_\pi \rightarrow (M^* \otimes d\pi)^*$ INDUCED BY THE PAIRING IS 1-1, SUPPOSE THAT $\langle M^* \otimes dx, f \rangle = 0$ FOR ALL $M^* \otimes x$, I.E. $\langle M^*, f(dx) \rangle = 0$: THEN WE MUST HAVE $f(dx) = 0$, ALL x , I.E. $f = 0$.

TO SEE THAT $\text{Hom}_\pi \rightarrow (M^* \otimes d\pi)^*$ IS ONTO, LET $L: M^* \otimes d\pi \rightarrow F$ BE F -LINEAR. FOR EACH $r \in d\pi$ WE GET A LINEAR MAP $M^* \rightarrow F$ GIVEN BY $L(M^* \otimes r)$, SO THERE IS A UNIQUE $m_r \in M$ SUCH THAT $\langle M^*, m_r \rangle = L(M^* \otimes r)$, NOTE THAT, FOR $x \in \pi$,

$\langle M^*, m_{xr} \rangle = L(M^* \otimes xr) = L(M^* x \otimes r) = \langle M^* x, r \rangle = \langle M^*, x m_r \rangle$, I.E. $m_{xr} = x m_r$. THUS, IF WE PUT $f(r) = m_r$, f IS A F -MODULE MAP $d\pi \rightarrow M$, AND CLEARLY f INDUCES THE GIVEN LINEAR FUNCTIONAL L ON $M^* \otimes d\pi$.

III - Local Homology/Cohomology

13,

A. LET X BE A CELL COMPLEX AND \tilde{X} THE UNIVERSAL COVERING COMPLEX.
 LET $\pi = \pi_1(X)$; THEN π ACTS FREELY ON \tilde{X} , SO THE CHAIN COMPLEX $C_*(\tilde{X})$ BECOMES A COMPLEX OF FREE $\mathbb{Z}\pi$ -MODULES. THE LOCAL HOMOLOGY OF X WITH COEFFICIENTS IN A π -MODULE M IS BY DEFINITION THE HOMOLOGY OF $C_*(\tilde{X}) \otimes_{\pi} M$; COHOMOLOGY LIKEWISE COMES FROM $\text{Hom}_{\pi}(C_*(\tilde{X}), M)$. THESE GROUPS CAN BE COMPUTED DIRECTLY FROM THE ORIGINAL COMPLEX X IF ONE USES THE RIGHT BOUNDARY/COBOUNDARY OPERATORS. FOR EXAMPLE, $C_*(\tilde{X})$ HAS A $\mathbb{Z}\pi$ -GENERATOR FOR EVERY CELL IN X , BUT THE FACES OF THIS CELL MAY ACQUIRE A π -COEFFICIENT BECAUSE THEY INVOLVE A DIFFERENT LIFTING THAN THE CHOSEN ONE. FOR EXAMPLE, SUPPOSE X HAS A CLOSED EDGE  REPRESENTING $p \in \pi$. THE

BOUNDARY $\partial \tilde{E}$ IS NOT $\tilde{U} - \tilde{U}$, BUT $p\tilde{U} - \tilde{U}$; THE TWO ENDS OF \tilde{E} DIFFER BY THE DECK TRANSLATION p . TO BE MORE PRECISE, CHOOSE LIFTINGS $\tilde{\sigma} \in \tilde{X}$ FOR EACH CELL $\sigma \in X$; IF THE BOUNDARY $\partial \sigma$ DOWNSTAIRS IS $\sum \neq \sigma_i$, THEN $\partial \tilde{\sigma}$ UPSTAIRS WILL BE VARIOUS TRANSLATES $\sum \neq p_i \tilde{\sigma}_i$, WITH $p_i \in \pi$. TO SIMPLIFY THE NOTATION, WE JUST REPLACE $\tilde{\sigma}_i$ BY σ_i EVERYWHERE AND GET THAT $C_*(\tilde{X})$ IS A BUNCH OF $\mathbb{Z}\pi$ -MODULES WITH BASIS = CELLS OF X , BUT THE BOUNDARY FACES ARE GIVEN WITH π -COEFFICIENTS.

THESE COEFFICIENTS p_i CAN FREQUENTLY BE DETERMINED BY OBSERVATION, AND HAVING DONE THIS, THE PROCESS OF COMPUTING THE HOMOLOGY OF $C_*(\tilde{X}) \otimes_{\pi} M$ OR $\text{Hom}_{\pi}(C_*(\tilde{X}), M)$ CAN PROCEED DIRECTLY.

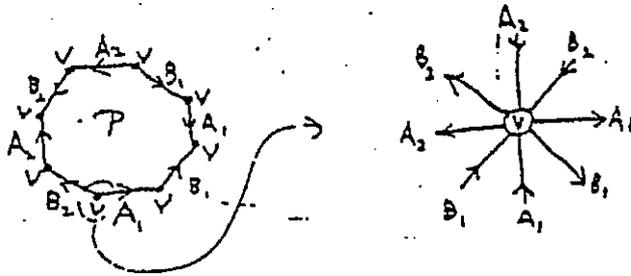
B. WE PRODUCE, AS A USEFUL EXAMPLE, THE STANDARD COMPLEX $C_*(\tilde{K}_g)$ FOR $K_g =$ A SURFACE OF GENUS g . THE CELLS OF K_g ARE:

0) ONE 0-CELL V

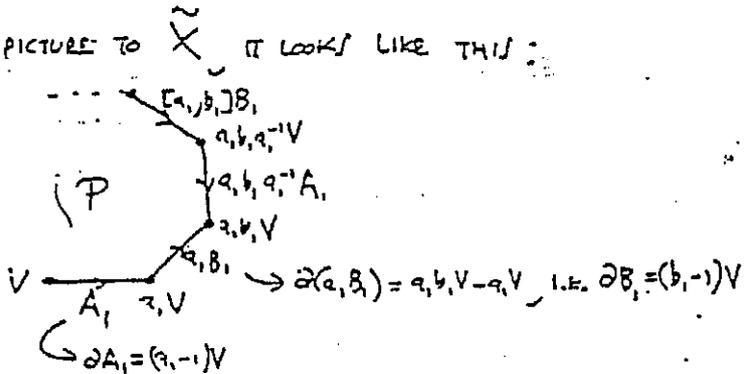
1) 2g ~~1-cell~~ A_i, B_i IN THE USUAL WAY TO GIVE GEOMETRIC GENERATORS
 $\therefore a_i, b_i \in \pi$

14,

2) ONE 2-CELL P



IF WE LIFT THE WHOLE PICTURE TO \tilde{X} IT LOOKS LIKE THIS:



HENCE THE BOUNDARY 'MAPS' ARE GIVEN BY:

$$\begin{aligned} \partial V &= 0 \\ \partial A_i &= (a_i - 1)V \quad \text{GENERALLY } \partial X = (x - 1)V \text{ FOR } X = A_i, B_i \\ \partial P &= A_1 - a_1 B_1 - a_1 b_1 a_1^{-1} A_1 - [a_1, b_1] B_1 + \dots \end{aligned}$$

WE CAN GIVE A COMPACT DESCRIPTION OF THIS COMPLEX IF WE REPLACE THE SYMBOLS A_i , ETC. BY da_i , ETC., SO THAT $C_1 = \text{FREE } \mathbb{Z}\pi\text{-MODULE ON } A_i, B_i$ BECOMES JUST $\mathbb{Z}\pi \otimes d\pi$, WHERE π IS THE FREE GROUP ON a_i, b_i , I.E. $\pi = \langle \pi \rangle$ (PUNCTURED SURF). THE BOUNDARY ∂P BECOMES, IN THIS NOTATION, $da_1 + a_1 db_1 - a_1 b_1 a_1^{-1} da_1 - \dots$, WHICH IS JUST $d\theta$, WHERE θ IS THE ATTACHING MAP AND THE STANDARD RELATION IN π . IF WE ALSO REPLACE C_0 BY $\mathbb{Z}\pi$ WITH 1 REPRESENTING V AND C_2 BY $\mathbb{Z}\pi$ WITH 1 REPRESENTING P , THE ABOVE COMPLEX BECOMES OUR STANDARD ONE FOR K_g :

$$0 \rightarrow \mathbb{Z}\pi \xrightarrow{-d\theta} \pi \otimes d\pi \xrightarrow{\epsilon} \mathbb{Z}\pi \rightarrow 0$$

WHERE ϵ , RECALL, SENDS dx TO $x - 1$, AND $\pi \otimes d\pi$ IS SHORTHAND FOR $\mathbb{Z}\pi \otimes d\pi$.

C. WE EASILY GENERALIZE THE ABOVE TO FIND $C_*(X)$ FOR ANY 2-COMPLEX X HAVING ONLY ONE VERTEX V . LET THE 1-CELLS BE A_i , REPRESENTING A BASIS OF THE FREE GROUP $F = \pi_1(1\text{-SKELETON})$, AND THE 2-CELLS BE P_k ATTACHED BY WORDS r_k IN THE A_i 'S. WE GET THEN A PRESENTATION $R \xrightarrow{j} F \xrightarrow{\pi} \pi_1$ FOR $\pi = \pi_1$ OF THE 2-COMPLEX K , WHERE R IS THE FREE GROUP ON THE P_k AND $j(P_k) = r_k$. SO FOR $C_*(K)$ WE HAVE $C_0 = \mathbb{Z}\pi$, $C_1 = \mathbb{Z}F$, $\partial_0 = \epsilon$, AS BEFORE; NOW HOWEVER, C_2 IS \mathbb{Z} -FREE ON MORE GENERATORS, ONE FOR EACH P_k AND THE CORRESPONDING BOUNDARY WILL BE dr_k , JUST AS WITH A SURFACE. WE CAN THEN USE $\mathbb{Z}R$ FOR C_2 , WITH GENERATORS dP_k , AND THE BOUNDARY MAP $dP_k \rightarrow \mathbb{Z}F$ IS JUST WHAT WE HAVE BEEN CALLING dj . SO WE GET

$$C_*(\tilde{K}) : 0 \rightarrow \mathbb{Z}R \xrightarrow{dj} \mathbb{Z}F \xrightarrow{\epsilon} \mathbb{Z}\pi \rightarrow 0$$

D. FINALLY WE PRODUCE A LOCAL COMPLEX FOR AN ORIENTABLE 3-MANIFOLD W , GIVEN AS A HANDLE DECOMPOSITION WITH ONE 0-HANDLE V ; g 1-HANDLES A_i ; g 2-HANDLES P_k AND ONE 3-HANDLE U . PUT $H = V \cup \{A_i\}$, A GENUS g HANDLE BODY, AND $K = \partial H$; $F = \pi_1(H)$ IS FREE ON $\{A_i\}$. WE MAY ALSO ASSUME THAT THE 2-HANDLES ARE GLUED TO DISJOINT SCC'S IN K ; IN FACT, WE CAN CHOOSE A GEOMETRIC BASIS p_k, q_k OF CURVES (BASED AT V) ON K SUCH THAT THE 2-HANDLE P_k IS GLUED TO p_k (IN OTHER WORDS, THE HANDLE DECOMPOSITION IS A HEERGAARD DECOMPOSITION). THE CURVES p_k, q_k REPRESENT ELEMENTS $r_k(q_i), s_k(q_i)$ IN $F = \pi_1(H)$. HENCE $C_*(\tilde{W})$ LOOKS LIKE:

$$0 \rightarrow \mathbb{Z}\pi \xrightarrow{\partial_3} \mathbb{Z}R \xrightarrow{dj} \mathbb{Z}F \xrightarrow{\epsilon} \mathbb{Z}\pi \rightarrow 0 \quad \text{WHERE THE FIRST TERM IS } C_3, \text{ GENERATORS: } i = U, \text{ AND THE REST IS OUR STANDARD 2-COMPLEX, WITH } R \text{ FREE ON } P_k \text{ AND } dj(\pm P_k) = dr_k. \text{ ONLY } \partial_3 \text{ IS YET TO BE DETERMINED. (} i = -\pi_1 = \pi_1(W) \text{ HERE)}$$

To do this, note that the curve γ_k intersects P_k once geometrically
 hence ~~and that~~ the front face of P_k is S_k times the back face. But
 ∂U hits both front and back face of each 2-handle (with opposite orientations)
 hence we get $\partial U = \sum (S_k - 1) \partial P_k$ and finally ~~and finally~~

$$C_*(\tilde{W}): 0 \rightarrow \mathbb{Z} \xrightarrow{1 - \sum S_k} \mathbb{Z} \xrightarrow{\partial P_k} \mathbb{Z} \xrightarrow{\partial F} \mathbb{Z} \rightarrow 0$$

E. We do the canonical example 3-manifold: a lens space $L(m, n)$, with $\pi_1 = \mathbb{Z}_m$.
 This has a handle decomposition with one handle of each dimension, and
 the 2-handle P is glued to the curve $p = A^m B^n$ in $K = \text{topus boundary}$
 of H , where $B = 0$ in $\pi_1 H$, A generates $\pi_1 H = \mathbb{Z}_m$. All our chain
 groups are of rank 1, with $\mathbb{Z}R$ generated by ∂P and $\mathbb{Z}F$ generated
 by ∂A . To get the γ -curve, choose x, y such that $\begin{vmatrix} mx & ny \\ -y & x \end{vmatrix} = 1$,
 then $\gamma = A^x B^y$ intersects p geometrically once.

Also p, γ represent $r = A^m$ and $s = A^x$ in F so our complex
 is

$$0 \rightarrow \mathbb{Z} \xrightarrow{1 - (A^x - 1) \partial P} \mathbb{Z} \xrightarrow{\partial P} \mathbb{Z} \xrightarrow{\partial A} \mathbb{Z} \rightarrow 0$$

or:

$$0 \rightarrow \mathbb{Z} \xrightarrow{1 - A^x} \mathbb{Z} \xrightarrow{\frac{\partial A^x}{\partial A}} \mathbb{Z} \xrightarrow{1 - A^{-1}} \mathbb{Z} \rightarrow 0$$

and note that $\frac{\partial (A^m)}{\partial A} = 1 + A + \dots + A^{m-1}$

If we tensor with \mathbb{Z} (π acting trivially), we get $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$
 and the usual homology $H_3 = \mathbb{Z}, H_2 = 0, H_1 = \mathbb{Z}_m, H_0 = \mathbb{Z}$.

If instead we tensor with \mathbb{C} , thought of as a π -module by
 $A = \text{multiplication by } e^{\frac{2\pi i}{m}}$, we get $A^x - 1 \neq 0$ (since $(m, x) = 1$),
 $\rightarrow \frac{\partial A^x}{\partial A} = 0, A^{-1} \neq 0$, and so the complex is

$$0 \rightarrow \mathbb{C} \xrightarrow{1 - A^x} \mathbb{C} \xrightarrow{\frac{\partial A^x}{\partial A}} \mathbb{C} \rightarrow 0, \text{ i.e. } H_* \text{ is acyclic}$$

for these twisted coefficients (not even a top or bottom class!). It is
 just this situation in which we are interested: when we have
 acyclic cohomology, we can define Reidemeister torsion.

(Co) Homology of a space Y . For simplicity we use a coefficient \mathbb{Z} .
 \mathbb{Z} -module M which is a vector space over some field F .

1) THE EULER CHARACTERISTIC $\sum (-1)^j \dim_F H_j(Y, M) = \chi(Y) \cdot \dim_F M$
 WHERE $\chi(Y)$ IS THE USUAL EULER CHARACTERISTIC. (THIS FOLLOWS BY COMPUTING
 π AT THE CHAIN LEVEL). IN PARTICULAR, $H_*(Y, M)$ CAN NOT BE ACYCLIC
 IF $\chi(Y) \neq 0$ — E.G., A SURFACE (EXCEPT T^2).

2) IF THERE IS A π -INVARIANT BILINEAR MAP $M \otimes M \rightarrow N$
 (E.G. $\langle x, y \rangle = \langle y, x \rangle$) THEN A CUP PRODUCT IS DEFINED
 ON LOCAL COHOMOLOGY $H_*^{\pi}(Y, M) \otimes H_*^{\pi}(Y, N)$ WITH THE USUAL
 PROPERTIES. IN PARTICULAR, IF \langle, \rangle HAS \pm SYMMETRY, THEN
 $\beta \cup \alpha = \pm (-1)^{\dim \alpha \cdot \dim \beta} \alpha \cup \beta$.

3) IF Y IS A COMPACT MANIFOLD AND $M \otimes M \xrightarrow{\langle, \rangle} F$ IS \mathbb{Z} -
 SYMMETRIC AND NONDEGENERATE, THEN \cup GIVES A POINCARÉ DUALITY
 ON $H^*(Y, M)$

THESE FACTS WILL BE USEFUL IN INVESTIGATING THE REPRESENTATION
 SPACES OF π_1 (SURFACE) INTO A LIE GROUP.

IV. LOW DIMENSIONAL COHOMOLOGY OF GROUPS

18.

LET π BE A GROUP AND M A π -MODULE; THE STANDARD DEFINITION OF $H^*(\pi, M)$ IS THAT OF THE COMPLEX $C^*(\pi, M)$, WHERE

$C^n(\pi, M) = \text{FUNCTIONS } \pi^n \rightarrow M$, WITH $C^0 = M$, AND

$$d^n f(x_1, \dots, x_{n+1}) = x_1 f(x_2, \dots, x_{n+1}) - f(x_1, x_2, \dots, x_{n+1}) + f(x_1, x_2, x_3, \dots, x_{n+1}) - \dots \pm f(x_1, x_2, \dots, x_n)$$

IN PARTICULAR: $d_0: C_0 = M \rightarrow C_1$ IS GIVEN BY $d_0(m): \pi \rightarrow M$ BY $x \rightarrow (x-1)m$.

AND $d_1: C_1 \rightarrow C_2$ BY $d_1 f(x_1, x_2) = x_1 f(x_2) - f(x_1, x_2) + f(x_1)$

HENCE:

$$Z^1 = X \text{HOM}(\pi, M), \quad B^1 = \text{ALL PRINCIPAL CROSSED FORMS,}$$

$$H^1 = \text{CROSSED MOD PRINCIPALS}$$

ALSO, $H^0 = Z^0 = \text{ALL } m \in M \text{ SUCH THAT } (x-1)m = 0, \text{ ALL } x \in \pi$ — I.E.,

m IS INVARIANT IN M UNDER THE ACTION OF π .

THE ABOVE COCHAIN COMPLEX IS JUST THE SAME AS THE LOCAL COHOMOLOGY OF A PARTICULAR $K(\pi, 1)$ COMPLEX, USING M COEFFICIENTS.

NOTE, HOWEVER, THAT ANY $K(\pi, 1)$ WILL DO JUST AS WELL. IN PARTICULAR, ANY SURFACE (CLOSED OR NOT) IS A $K(\pi, 1)$, SO WE CAN USE OUR STANDARD SURFACE COMPLEX TO CALCULATE $H^*(\text{SURFACE}, M)$.

BECAUSE OF THIS, WE WILL FREQUENTLY IDENTIFY $H^*(\text{SURFACE})$ AND

$H^*(\pi, (\text{SURFACE}))$. IN THE SAME WAY, FOR A ^{3-dim} HANDLEBODY H_g OF GENUS g ,

WE WILL IDENTIFY $H^*(\pi, (H_g))$ AND $H^*(H_g)$. THE LATTER CAN BE

EASILY COMPUTED FROM A 1-COMPLEX FOR H_g — NAMELY $0 \rightarrow \pi \xrightarrow{\epsilon} \mathbb{Z}\pi \rightarrow 0$
($\pi = \pi_1 H_g$)

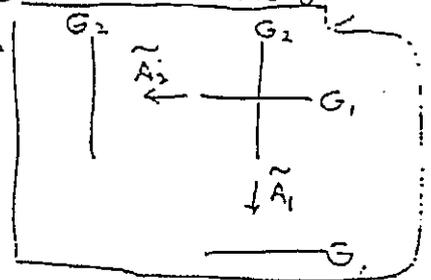
V. REPRESENTATION SPACES $\text{Hom}(\pi, G)$ FOR $G = \text{LIE GROUP}$ 19.

A. LET π BE A DISCRETE GROUP AND G A LIE GROUP; WE DENOTE BY R_π THE SPACE $\text{Hom}(\pi, G)$ OF HOMOMORPHISMS OF π INTO G . LET L BE THE LIE ALGEBRA OF G . AT A GIVEN POINT ρ OF R_π , I.E. A REP $\rho: \pi \rightarrow G$, L ACQUIRES THE STATUS OF A LEFT π -MODULE WITH THE ACTION OF $\alpha \in \pi$ ON $l \in L$ GIVEN BY $\alpha l = \text{Ad}_{\rho(\alpha)} l = \rho(\alpha) l \rho(\alpha)^{-1}$. AS ρ MOVES AROUND IN R_π , THIS ACTION MOVES ALSO, SO WE MAY WRITE THE MODULE AS L_ρ TO MAKE THE ACTION SPECIFIC AND EVIDENT; THE UNDERLYING VECTOR SPACE REMAINS L , OF COURSE. THE DUAL OF THE LIE ALGEBRA L^* BECOMES THEN A RIGHT π -MODULE L_ρ^* . THERE IS A STANDARD Ad -INVARIANT (AND HENCE ALSO π -INVARIANT FOR ANY $\rho \in R_\pi$) NONDEGENERATE SYMMETRIC BILINEAR FORM $\langle \cdot, \cdot \rangle$ ON L , THE KILLING FORM, WITH WHICH WE MAY IDENTIFY L_ρ AND L_ρ^* .

B. IF $\alpha \in \pi$ WE GET A G -VALUED FUNCTION $\tilde{\alpha}: R_\pi \rightarrow G$ GIVEN BY $\tilde{\alpha}(\rho) = \rho(\alpha)$, AND WE HAVE $\tilde{\alpha}xy = \tilde{\alpha}x \tilde{\alpha}y$. FOR F A FREE GROUP ON $\{A_i\} (i=1, \dots, m)$ WE MAY IDENTIFY R_F WITH G^m BY THE MAP $\rho \rightarrow (\rho(A_1), \dots, \rho(A_m))$, I.E. $\rho \rightarrow (\tilde{A}_1, \dots, \tilde{A}_m)(\rho)$: THE \tilde{A}_i ARE G -VALUED "COORDINATES". WE CAN GET A BASIS FOR THE COTANGENT SPACE T^*R_F AT ρ , DENOTED T^*F_ρ , AS FOLLOWS: $d\tilde{A}_i: T^*F \rightarrow T^*G$

AND $d\tilde{A}_i \cdot \tilde{A}_i^{-1}$ IS AN L -VALUED 1-FORM ON R_F . THIS FORM KILLS THE TANGENTS TO EVERY FACTOR OF $G \times \dots \times G$ EXCEPT FOR THE i TH

AND MAPS THAT ISOMORPHICALLY ONTO L :
 HENCE, IF l_j^* IS A BASIS FOR L^* , THEN
 $\langle l_j^*, d\tilde{A}_i \cdot \tilde{A}_i^{-1} \rangle$ ($i=1, \dots, m; j=1, \dots, \dim G$)



ARE ORDINARY 1-FORMS ON R_F AND CLEARLY FORM A BASIS FOR THE COTANGENT SPACE T^*F_ρ AT ANY POINT $\rho \in R_F$. NOTE THAT $\dim R_F = \dim T^*F_\rho = m \cdot \dim G$.

C. LET NOW π BE A FINITELY PRESENTABLE GROUP WITH PRESENTATION 20.

IN THE USUAL FORM $R \xrightarrow{\beta} F \xrightarrow{p} \pi \rightarrow 0$: F IS FREE ON A_i ($i=1, \dots, m$)
 R IS FREE ON R_j ($j=1, \dots, n$). WE HAVE AN INDUCED SEQUENCE OF SMOOTH
 MAPS $R_R \xrightarrow{g_*} R_F \xrightarrow{p_*} R_\pi \rightarrow 0$ WHICH IS "EXACT" IN THE SENSE THAT
 g_* IS AN IMBEDDING AND ITS IMAGE EQUALS "KEY g_* ", I.E., π EQUALS
 g_*^{-1} (TRIVIAL HON IN R_R) - WITH THIS IDENTIFICATION WE WILL TREAT
 R_π AS BEING CONTAINED IN R_F . WE WANT TO SEE R_π AS
 A SINGULAR MANIFOLD, WHICH PROMPTS THE

DEFINITION: $p \in R_\pi$ IS NONSINGULAR IF \exists A NBHD U OF p IN R_F
 SUCH THAT RANK OF $d(g_*)$ IS CONSTANT ON U .

WE QUOTE THE FOLLOWING STANDARD THEOREM:

IF $f: M \rightarrow N$ IS SMOOTH AND RANK(df) IS CONSTANT ON A NBHD U
 OF $m \in M$, THEN $f^{-1}(f(m)) \cap U$ IS A SMOOTH SUB-MANIFOLD THROUGH m OF DIMENSION
 $\dim M - \text{RANK } df$.

APPLYING THIS TO THE CASE AT HAND:

PROP^o IF p IS NONSINGULAR IN R_π THEN R_π IS LOCALLY (NEAR p) A
 SMOOTH SUBMANIFOLD OF R_F OF CODIMENSION RANK(g_*). (IN MOST, BUT
 MAYBE NOT ALL, CASES: FOR US!!!, $m = \#$ GENERATORS WILL BE $\geq n = \#$ RELATIONS
 AND RANK df WILL BE n AT NONSINGULAR p 'S.) WE DENOTE THE
 MANIFOLD OF NONSINGULAR POINTS IN R_π BY R_π^* . AT A POINT p
 OF R_π^* , THEN, WE CAN FIND THE TANGENT AND COTANGENT SPACE OF R_π^* .
 IN FACT, JUST BY DEFINITION OF THESE, WE HAVE EXACT SEQUENCES

$$\begin{array}{ccccccc} TR & \xleftarrow{d(g_*)} & TF & \xleftarrow{d(p_*)} & T\pi & \rightarrow & 0 \\ \overleftarrow{TR} & \xrightarrow{d(g_*)} & \overleftarrow{TF} & \xrightarrow{d(p_*)} & \overleftarrow{T\pi} & \rightarrow & 0 \end{array}$$

D. LET NOW $x \in \pi$ AND $p \in R_\pi$. JUST AS WITH A FREE GROUP, 21.
 \tilde{x} DEFINE A SMOOTH MAP $\tilde{x}: R_\pi \rightarrow G$ AND INDUCES AN
 L -VALUED COTANGENT $\omega_x = d\tilde{x} \cdot \tilde{x}^{-1}$ (ANY p). ALSO, FOR $l^* \in L^*$,
 $\langle l^*, \omega_x \rangle$ IS AN ORDINARY COTANGENT AT p . WE GET:

$$\omega_{xy} = d(\tilde{xy}) \tilde{xy}^{-1} = d(\tilde{x}\tilde{y}) \tilde{y}^{-1}\tilde{x}^{-1} + \tilde{x} d\tilde{y} \tilde{y}^{-1}\tilde{x}^{-1} = \omega_x + \tilde{x} \omega_y \tilde{x}^{-1}$$

$$= \omega_x + x \omega_y \text{ USING THE } \pi\text{-ACTION OF } x \text{ ON } \omega_y \in L = L_p.$$

HENCE THE ASSIGNMENT $x \rightarrow \omega_x$ GIVES A COISSUED HOM $\pi \rightarrow L_p \otimes_{\mathbb{T}\pi_p}^* \mathbb{T}\pi_p$
 AND HENCE WE HAVE AN INDUCED π -MAP $d\pi \rightarrow L_p \otimes_{\mathbb{T}\pi_p}^* \mathbb{T}\pi_p$. ALSO, WE
 GET A MAP $L_p^* \otimes_{\mathbb{T}\pi_p}^* \mathbb{T}\pi_p \rightarrow \mathbb{T}\pi_p$ BY $l^* \otimes dx \rightarrow \langle l^*, \omega_x \rangle$.

WE CLAIM THAT:

PROP. 9

$$L_p^* \otimes_{\mathbb{T}\pi_p}^* \mathbb{T}\pi_p \rightarrow \mathbb{T}\pi_p \text{ IS AN ISOMORPHISM}$$

PROOF: WE START BY ~~PROVING~~ PROVING IT FOR FREE GROUPS, E.G., F .

IN THIS CASE, $L_p^* \otimes_{\mathbb{T}F_p}^* \mathbb{T}F_p$ HAS THE BASIS $l_j^* \otimes dA_i$, WHICH MAP BY
 DEFINITION TO THE COTANGENTS $\langle l_j^*, dA_i \cdot A_i^{-1} \rangle$. BUT WE HAVE
 ALREADY SEEN, THAT THE LATTER FORM A BASIS FOR $\mathbb{T}F$ AT p .

THE SAME IS TRUE FOR $\mathbb{T}R$, AND WE HAVE NOW A DIAGRAM

$$\begin{array}{ccccc} L^* \otimes_{\mathbb{T}R}^* \mathbb{T}R & \xrightarrow{d\pi} & L^* \otimes_{\mathbb{T}F}^* \mathbb{T}F & \xrightarrow{d\pi} & L^* \otimes_{\mathbb{T}\pi}^* \mathbb{T}\pi \rightarrow 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \\ \mathbb{T}R & \xrightarrow{d\pi} & \mathbb{T}F & \xrightarrow{d\pi} & \mathbb{T}\pi \rightarrow 0 \end{array}$$

AND AN EASY DIAGRAM CHASE SHOWS THAT IT COMMUTES. HENCE THE
 FINAL VERTICAL MAP IS AN ISOMORPHISM ALSO.

DUALIZING THE ABOVE DIAGRAM AND USING PROPOSITION 7, WE GET

$$\begin{array}{ccccc} \mathbb{T}R & \xleftarrow{d\pi} & \mathbb{T}F & \xleftarrow{d\pi} & \mathbb{T}\pi \leftarrow 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \text{Hom}_{\mathbb{T}R}(\mathbb{T}R, L) & \xleftarrow{d\pi^*} & \text{Hom}_{\mathbb{T}F}(\mathbb{T}F, L) & \xleftarrow{d\pi^*} & \text{Hom}_{\mathbb{T}\pi}(\mathbb{T}\pi, L) \leftarrow 0 \end{array}$$

~~WE~~ ^{USUALLY} WILL MAKE THESE IDENTIFICATIONS IN THE FOLLOWING LECTURE.

As a corollary, we have:

PROPOSITION $p \in R_m$ is nonsingular if $g^* = \text{Hom}(dF, L_p) \rightarrow \text{Hom}(dR, L_p)$
is surjective. In this case $\dim R_p$ at p is $(m-n)\dim G$

PROOF: WE HAVE THEN THAT $TF_p \xrightarrow{dg} TR_p$ is surjective, and hence
also surjective in a nbhd U of p in R_F , so $\text{RANK } dg = m \cdot \dim G$ in U , and
the result follows by Prop. 8

E. THE IDENTIFICATION OF $T\pi_p$ WITH $\text{Hom}(d\pi, L_p) \cong X\text{HOM}(\pi, L_p)$

CAN BE MADE MORE CONCRETE BY THE FOLLOWING CONSIDERATIONS.

A TANGENT VECTOR AT $p \in R_m$ IS GIVEN BY THE DERIVATIVE $\left. \frac{d}{dt} \right|_{t=0}$ OF A CURVE
OF PTS p_t WITH $p_0 = p$. THIS TANGENT VECTOR APPLIED TO THE FUNCTION
 $\tilde{x} : R_m \rightarrow G$ WILL GIVE A TANGENT VECTOR TO G AT $p(x)$ AND
MOVING IT BACK TO THE IDENTITY BY RIGHT MULTIPLICATION BY $p(x)^{-1}$ GIVES

$$\left[\frac{d p_t(x)}{dt} \right]_{t=0} p(x)^{-1} \in L. \text{ CALL THIS } \varphi(x). \text{ THEN CALCULATING}$$

$\varphi(xy)$ GIVES THAT $\varphi(x) + x\varphi(y)$, I.E., $\varphi \in X\text{HOM}(\pi, L_p)$,
WHICH IS THE IDENTIFICATION WE DESIRE.

E. THE IDENTIFICATION OF $T_{\pi} \rho$ WITH $\text{Hom}(\mathfrak{d}\pi, L_p) \cong X\text{HOM}(\pi, L_p)$ CAN BE MADE MORE CONCRETE BY THE FOLLOWING CONSIDERATIONS.

A TANGENT VECTOR AT $\rho \in R_{\pi}^*$ IS GIVEN BY THE DERIVATIVE AT $t=0$ OF A CURVE OF REPS ρ_t WITH $\rho_0 = \rho$. THIS TANGENT VECTOR APPLIED TO THE FUNCTION $\tilde{\chi}: R_{\pi} \rightarrow G$ WILL GIVE A TANGENT VECTOR TO G AT $\rho(x)$, NAMELY $\left. \frac{d\rho_t(x)}{dt} \right|_{t=0}$.

MOVING IT BACK TO THE IDENTITY OF G BY RIGHT TRANSLATION GIVES $\varphi(x) = \left. \frac{d\rho_t(x)}{dt} \right|_{t=0} \rho(x)^{-1} \in L$. CALCULATING $\varphi(xy)$ GIVES

$\varphi(x) + x \varphi(y)$, I.E., $\varphi \in X\text{HOM}(\pi, L_p)$, WHICH IS THE IDENTIFICATION WE DESIRE.

VI. CONTIGUOUS CLASS SPACES.

24.

A. LET G BE AS BEFORE. TWO REPS $\rho_i \in R_\pi$ ARE CONJUGATE

IF $\exists g \in G$ SUCH THAT $\rho_2(x) = g \rho_1(x) g^{-1}$ FOR ALL $x \in \pi$;

WE WRITE $\rho_2 = g \rho_1 g^{-1}$. IN THIS WAY WE GET A SET OF

CONJUGACY CLASSES OF REPRESENTATIONS. WE WOULD LIKE TO MAKE

THIS SET INTO SOME KIND OF SINGULAR MANIFOLD, BUT PROBLEMS

ARISE WHEN G IS NON-COMPACT, EVEN IN THE CASE OF $SL(2, \mathbb{R})$:

THE QUOTIENT TOPOLOGY IS NON-HAUSDORFF. FOR EXAMPLE, IF $\pi = \mathbb{Z}$

THEN $R_\pi = SL(2, \mathbb{R})$ SO WE ARE LOOKING AT CONJUGACY CLASSES

IN $SL(2, \mathbb{R})$. NOW $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ AND $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ARE NOT CONJUGATE, BUT,

FOR ANY $x \neq 1$, $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ AND $\begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}$ ARE CONJUGATE. HENCE

AS $x \rightarrow 1$, THEIR CONJUGACY CLASS SETS CLOSE AND CLOSED TO

BOTH $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ AND $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, I.E. EVERY OPEN SET CONTAINING $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

INTERSECT EVERY OPEN SET CONTAINING $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$.

THE WAY THIS IS HANDLED IS BY QUOTIENTING DOWN R_π

A BIT FURTHER (IN THE EXAMPLE ABOVE, BY IDENTIFYING THE TWO

"BAD" CONJUGACY CLASSES $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ AND $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$, AS WELL AS SOME OTHERS).

SOME INFORMATION IS LOST IN THIS PROCESS, AND IT CAUSES NUMEROUS

PROBLEMS, BUT THERE IS NO REAL (I.E. HAUSDORFF) ALTERNATIVE.

TO SEE HOW IT IS DONE, WE RESTRICT OUR ATTENTION NOW

AND HENCEFORTH TO LINEAR GROUPS (G) , I.E. $G =$ A GROUP OF

MATRICES, WHICH WE REQUIRE TO ACT IRREDUCIBLY ON ^{THE UNDERLYING} VECTOR SPACE V .

UNLESS OTHERWISE EXPLICITLY MENTIONED, V WILL ALWAYS BE A COMPLEX

VECTOR SPACE. WE HAVE THEN THE TRACE FUNCTION $\text{Tr}: G \rightarrow \mathbb{C}$

WHICH IS INVARIANT UNDER CONJUGATION, AND GIVEN ANY $x \in \pi$

WE HAVE THE COMPOSITION $\text{Tr} \circ x: R_\pi \xrightarrow{\cong} G \xrightarrow{\text{Tr}} \mathbb{C}$ IS INVARIANT

UNDER THE CONJUGATION ACTION ~~of~~ of G ON R_π . FOR

FINITELY GENERATED π ~~IT~~ IT TURNS OUT, ~~IF~~ IF WE DEFINE R_π^{**}

TO BE THE SET OF IRREDUCIBLE REPS IN R_π , THAT:

a) \exists A FINITE SET $\chi_1, \dots, \chi_k \in \pi$. SUCH THAT $\rho_1, \rho_2 \in \hat{R}_\pi^{**}$
 ARE CONJUGATE REPS IFF $\text{Tr}_{\chi_i}(\rho_1) = \text{Tr}_{\chi_i}(\rho_2)$ FOR ALL i } 5.

b) DEFINING $T = (\text{Tr}_{\chi_1}, \dots, \text{Tr}_{\chi_k}) : R_\pi \rightarrow \mathbb{C}^k$, THEN
 $T(R_\pi^{**})$ IS A SMOOTH SUBMANIFOLD OF \mathbb{C}^k , WHICH WE CALL \hat{R}_π^{**} .

c) T RESTRICTED TO \hat{R}_π^{**} IS A PRINCIPAL BUNDLE PROJECTION WITH
 FIBER AND GROUP $G/Z(G)$, WHERE $Z(G)$ IS THE CENTER OF G .

NOTE THAT BY a), EACH FIBER CONSISTS OF EXACTLY ONE
 CONJUGACY CLASS. BY b), THE CONJUGACY CLASS IS $\simeq G/Z(G)$,

I.E. THE ONLY ELEMENTS OF G COMMUTING WITH THE REPRESENTATION
 ARE THE CENTRAL ELEMENTS OF G .

B. IN MOST OF THESE LECTURES (EXCEPT FOR SOME ILLUSTRATIVE EXAMPLES)
 G WILL HAVE FINITE CENTER (E.G., $SL(n, \mathbb{C})$), SO $\dim G/Z(G) = \dim G$.
 THUS THE TWO GROUPS HAVE THE SAME LIE ALGEBRA L , AND SO THE
 TANGENT SPACE TO THE FIBER AT ANY $\rho \in \hat{R}_\pi^{**}$ HAS A CANONICAL
 IDENTIFICATION WITH L VIA THE CONJUGATION ACTION. NOW THE TANGENT
 SPACE AT $\hat{\rho} \in \hat{R}_\pi^{**}$ IS IDENTIFIED WITH $TR_\pi / T(\text{FIBER})$, AND
 WE HAVE ALREADY IDENTIFIED TR_π WITH $X\text{Hom}(\pi, L_\rho)$, WHERE
 A TANGENT VECTOR TO A CURVE OF REPRESENTATIONS ρ_t IS GIVEN
 BY THE CROSSED HOM $\varphi(x) = \left. \frac{d\rho(x)}{dt} \right|_{t=0} \rho(x)^{-1}$ ($x \in \pi$).

TO CARRY OUT THIS IDENTIFICATION FOR TANGENTS TO THE FIBER,
 LET g_t BE A CURVE IN G WITH $g_0 = 1$ AND $\left. \frac{dg}{dt} \right|_{t=0} = l \in L$.

THE CURVE OF CONJUGATE REPS $\rho_t = g_t \rho g_t^{-1}$ THROUGH ρ GIVES THE
 CROSSED HOM

$$\varphi(x) = \frac{d(g_t(x)g_t^{-1})}{dt} \Big|_{t=1} (1 \cdot p(x) \cdot 1^{-1})^{-1}$$

$$= \left\{ \left[\frac{dg}{dt} \right]_0 p(x) \cdot 1^{-1} + 1 \cdot \frac{dp(x)}{dt} \cdot 1^{-1} - 1 \cdot p(x) \cdot 1^{-1} \left[\frac{dg}{dt} \right]_0 \cdot 1^{-1} \right\} p(x)^{-1} \quad 26.$$

$$= (l \cdot p(x) - p(x) \cdot l) p(x)^{-1} = l - p(x)l p(x)^{-1} = l - xl = (1-x)l$$

.. THUS, $\varphi(x)$ IS THE PRINCIPAL CROSSED HOM CORRESPONDING TO $-l$.

.. THIS MAKES EXPLICIT THE IDENTIFICATION OF TFIBER WITH L .

.. AND WE HAVE A SEQUENCE OF NATURAL ISOMORPHISMS

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{TFIBER} & \rightarrow & TR_{\pi} & \rightarrow & \widehat{TR}_{\pi} \rightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \rightarrow & \text{PRINC}(\pi, L_p) & \rightarrow & \text{XHOM}(\pi, L_p) & \rightarrow & \text{XHOM} \rightarrow 0 \\ & & & & \text{PRINC} & & \end{array}$$

AT IRREDUCIBLE ρ . BUT RECALL THAT FOR GROUP COHOMOLOGY,

$B^1(\pi, L_p) = \text{PRINCIPAL HOMS } \pi \rightarrow L_p$ AND

$Z^1(\pi, L_p) = \text{XHOM}(\pi, L_p)$, SO \widehat{TR}_{π} IS NATURALLY ISOMORPHIC

TO $H^1(\pi, L_p)$, AND THE ABOVE SEQUENCE READS

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{TFIBER} & \rightarrow & TR_{\pi} & \rightarrow & \widehat{TR}_{\pi} \rightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \rightarrow & B^1(\pi, L_p) & \rightarrow & Z^1(\pi, L_p) & \rightarrow & H^1(\pi, L_p) \rightarrow 0 \end{array}$$

WE MAKE FREQUENT USE OF THESE IDENTIFICATIONS.

REMARKS: WITH MORE TECHNICAL EFFORT, THE ABOVE CONSTRUCTION OF \widehat{R}_{π}

AND ITS TANGENT SPACE AT IRREDUCIBLE REPS COULD BE DONE FOR LINEAR ALGEBRAIC GROUPS OVER ANY FIELD F , SO THAT \widehat{R}_{π} BECOMES AN ALGEBRAIC VARIETY OVER F , NONSINGULAR FOR IRREDUCIBLES.

WE AVOID THIS FOR THE PRESENT, BUT MAY HAVE TO DO IT LATER WHEN WE GET TO THE RATIONALITY OF THE TORSION POLYNOMIAL

C. WE COLLECT HERE SOME BASIC FACTS ABOUT THE (CO)HOMOLOGY OF IRREDUCIBLE REPRESENTATIONS. 27.

1) THE PAIRING $H^1(\pi, L_\rho) \otimes H_1(\pi, L_\rho^*) \rightarrow F$ (WHERE $F = \mathbb{C}$ OR \mathbb{R} , DEPENDS ON THE LIE GROUP) SHOWS THAT, IF ρ IS IRREDUCIBLE,

$H^1(\pi, L_\rho)$ IS NATURALLY ISOMORPHIC TO BOTH $H_1(\pi, L_\rho^*)^*$ AND TO $\hat{H}_1(\pi, \mathbb{R})$. SO $H_1(\pi, L_\rho^*)$ IS NATURALLY ISOMORPHIC TO $\hat{H}_1(\pi, \mathbb{R})^*$.

2) **PROP. 12** IF ρ IS IRREDUCIBLE, AND V IS THE UNDERLYING SPACE ON WHICH G ACTS, THEN:

a) $H^0(\pi, V_\rho) = 0$

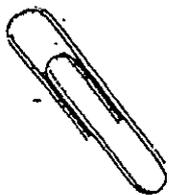
b) $H^0(\pi, L_\rho) = 0$ IF $Z(G)$ IS FINITE

PROOF: THE FIRST IS JUST BY DEFINITION: H^0 IS THE SPACE OF INVARIANTS, AND V HAS NO π -INVARIANT SUBSPACES. TO SEE THE SECOND, LET $l \in L$ BE INVARIANT, I.E. $\chi l = \rho(\chi) l \rho(\chi)^{-1} = l$ FOR ALL $\chi \in \pi$, AND LET $V_\lambda \subset V$ BE THE λ -EIGENSPACE OF l FOR SOME EIGENVALUE λ OF l . IF $\chi \in \pi$ AND $v \in V_\lambda$, THEN $\rho(\chi)v = \rho(\chi)lv = \lambda \rho(\chi)v$, SO $\rho(\chi)v \in V_\lambda$ ALSO, I.E.

$\rho(\chi)(V_\lambda) \subset V_\lambda$ FOR EVERY $\chi \in \pi$. HENCE V_λ IS AN INVARIANT SUBSPACE AND SO $V_\lambda = V$. THUS l MUST BE A SCALAR MATRIX λI

BUT THE ONLY SCALAR MATRIX IN L IS 0; OTHERWISE WE COULD EXPONENTIATE l AND GET A 1-DIMENSIONAL CENTRAL SUBGROUP OF G .

3) IF π IS A CLOSED SURFACE GROUP, THEN USING THE KILLING FORM ON L , THE CUP PRODUCT & POINCARÉ DUALITY GIVE A DUAL PAIRING $H^0(\pi, L_\rho) \otimes H^2(\pi, L_\rho) \rightarrow F$. IN PARTICULAR, IF ρ IS IRREDUCIBLE, THEN $H^0 = 0$, SO $H^2 = 0$. BY PROP. 11 AND THE NOTE FOLLOWING IT, ρ IS A NONSINGULAR REP IN \mathbb{R}_+



VII. VOLUMES

28

A. LET V BE A VECTORSPACE OF DIMENSION n OVER A FIELD F .

A VOLUME ON V IS A CHOICE OF A NONZERO ELEMENT θ IN $\Lambda^n V$.

A VOLUME CAN BE WRITTEN IN THE FORM $e_1 \wedge \dots \wedge e_n$ FOR SOME

BASE $\{e_i\}$ OF V ; IF θ IS SPECIFIED ON V , WE CALL ANY

SUCH BASIS A VOLUME BASIS OF V . IN THE SPECIAL CASE WHEN

$V=0$, WE DEFINE "VOLUME" TO MEAN A NON ZERO NUMBER IN F .

NOW LET $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ BE A SHORT EXACT SEQUENCE

OF VECTOR SPACES, AND SUPPOSE A, C HAVE VOLUMES

$\theta_A = a_1 \wedge \dots \wedge a_m$ AND $\theta_C = c_1 \wedge \dots \wedge c_n$. WE CAN LIFT c_j TO

ELEMENTS $b_j \in B$, WELL DEFINED MODULO LINEAR COMBINATIONS

OF THE a_i 'S, AND SO $\theta_B = a_1 \wedge \dots \wedge a_m \wedge b_1 \wedge \dots \wedge b_n$ IS INDEPEN-

DENT OF THE LIFTING. WE WRITE $\theta_B = \theta_A \theta_C$ TO INDICATE THE

DEPENDENCE OF θ_B ON θ_A, θ_C . SIMILARLY, IF θ_A, θ_B ARE

GIVEN, THERE IS A UNIQUE VOLUME θ_C ON C SUCH THAT $\theta_B = \theta_A \theta_C$,

AND LIKEWISE FOR θ_A , GIVEN θ_B AND θ_C . FOR EXAMPLE,

GIVEN $\theta_A = a_1 \wedge \dots \wedge a_m$ AND θ_B , EXTEND THE PARTIAL

BASE $\{a_i\}$ OF B TO A VOLUME BASIS $a_1, \dots, a_m, b_1, \dots, b_n$.

THEN THE PROJECTIONS OF $\{b_j\}$ TO C GIVE A BASIS DEFINING

THE REQUIRED θ_C . WE WRITE $\theta_C = \theta_B / \theta_A$ TO INDICATE THIS.

NOTE THAT FOR $f \in F$, THESE RELATIONSHIPS BEHAVE INTUITIVELY

WITH RESPECT TO MULTIPLICATION, E.G. $(f\theta_A)\theta_C = \theta_A(f\theta_C) = f(\theta_A\theta_C)$,

$$\theta_B / (f\theta_A) = \frac{1}{f} (\theta_B / \theta_A), \text{ ETC.}$$

AS A SPECIAL CASE, CONSIDER $C=0$ I.E. THE

3-TERM EXACT SEQUENCE $A \xrightarrow{f} B \rightarrow 0$. THEN

$\theta_C = \theta_B / \theta_A$ IS JUST THE RATIO OF θ_B AND $f(\theta_A)$, I.E.

A NON ZERO $\#$ IN F ; LIKEWISE FOR $0 \rightarrow A \xrightarrow{f} B$

GIVEN A VOLUME θ ON A THERE IS A CANONICAL

VOLUME ON A^* SUCH THAT UNDER THE INDUCED PAIRING

f

OF M_A AND M_{A^*} TO F WE HAVE $\langle \theta, \theta^* \rangle = 1$ 29
 IF $\theta = \alpha_1 \wedge \dots \wedge \alpha_m$, THEN θ^* IS ALSO GIVEN BY $\alpha_1^* \wedge \dots \wedge \alpha_m^*$,
 WHERE $\{\alpha_i^*\}$ IS THE DUAL BASIS OF A^* . NOTE THAT $(f\theta)^* = f^*\theta^*$.

NOW SUPPOSE $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ IS EXACT; THE DUAL
 SEQUENCE $0 \rightarrow C^* \rightarrow B^* \rightarrow A^* \rightarrow 0$ IS ALSO EXACT. GIVEN
 θ_A, θ_C ON A, C WE CAN CONSTRUCT $\theta_A \theta_C$ ON B AND $\theta_C^* \theta_A^*$
 ON B^* .

PROP. 13 $(\theta_A \theta_C)^* = (-1)^{\dim A \cdot \dim C} \theta_C^* \theta_A^*$

PROOF: LET $p = \dim A, q = \dim C$, CHOOSE VOLUME BASES
 $\{\alpha_i\}, \{\alpha_j^*\}$ FOR A, C AND LET THE IMAGE OF α_i IN B BE b_i ,
 LIFTS OF α_j TO B BE b_j^* , SO THAT $\theta_A \theta_C = b_1 \wedge \dots \wedge b_p \wedge b_{p+1} \wedge \dots \wedge b_{p+q}$.
 IN THE DUAL SEQUENCE $C^* \rightarrow B^* \rightarrow A^*$ WE HAVE ~~WITH THAT~~
 $\theta_C^* = \alpha_1^* \wedge \dots \wedge \alpha_q^*$, $\theta_A^* = \alpha_1^* \wedge \dots \wedge \alpha_p^*$, $(\theta_A \theta_C)^* = \dots$
 IT IS EASY TO SEE THAT b_j^* IS A LIFT OF α_j^* AND α_j^* COEF. D. b_j^* .
 SO $\theta_C^* \theta_A^* = b_1^* \wedge \dots \wedge b_p^* \wedge \alpha_1^* \wedge \dots \wedge \alpha_q^* = (-1)^{pq} (\theta_A \theta_C)^*$.

B. NOW LET $0 \rightarrow C_n \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_1} C_0 \rightarrow 0$ BE AN EXACT SEQUENCE
 OF VECTOR SPACES, EACH WITH VOLUME V_i . FROM
 $C_n \rightarrow C_{n-1} \rightarrow \partial C_{n-1}$ (WE FREQUENTLY LEAVE OUT LEADING & TRAILING 0'S
 OF SHORT EXACT SEQUENCES) WE GET A VOLUME V_{n-1}/V_n ON ∂C_{n-1} .
 AGAIN, FROM $\partial C_{n-1} \rightarrow C_{n-2} \rightarrow \partial C_{n-2}$, THE VOLUMES V_{n-1}/V_n AND V_{n-2}
 GIVE A VOLUME $V_{n-2}/(V_{n-1}/V_n)$ ON ∂C_{n-2} . CONTINUING IN THIS WAY GIVES A VOLUME
 $V_1/(V_2/(V_3/(\dots/V_n)))$ ON $\partial C_1 = C_0$. FINALLY, WE COMPARE
 THIS VOLUME WITH THE ORIGINAL VOLUME V_0 ON C_0 AND GET
 A NON ZERO NUMBER $V_0/(V_1/(\dots/V_n))$ IN F . THIS IS
 THE TORSION OF THE VOLUME EXACT SEQUENCE C_F , AND WE
 DENOTE IT BY $\text{CT}(C_F)$. IT IS EASY TO SEE THAT CHANGING V
 $(f_i \in F^*)$ WILL CHANGE IT TO $f_0 f_1^{-1} f_2 \dots f_n^{-1} \text{CT}$.

HERE IS AN ALTERNATE WAY TO CALCULATE TORSION WHICH IS FREQUENTLY USEFUL. CHOOSE ARBITRARY VOLUMES w_i FOR ∂C_i , $i=1, \dots, n-1$. WE HAVE THE $\int E \int$ 'S

$C_n \rightarrow C_{n-1} \rightarrow \partial C_{n-1}$; LET $T_{n-1} = \frac{v_n w_{n-1}}{v_{n-1}}$

$\partial C_{n-1} \rightarrow C_{n-2} \rightarrow \partial C_{n-2}$; LET $T_{n-2} = \frac{w_{n-1} w_{n-2}}{v_{n-2}}$

$\partial C_2 \rightarrow C_1 \rightarrow C_0$; LET $T_1 = \frac{w_2 v_0}{v_1}$

THEN $\tau(C_n) = T_1 T_2^{-1} \dots T_{n-1}^{(-1)^{n-1}}$

PROOF: THE FIRST VOLUME EQUATION MAY BE WRITTEN $v_{n-1} = v_n \left(\frac{w_{n-1}}{v_{n-1}} \right)$

SO $\frac{w_{n-1}}{v_{n-1}}$ IS THE INDUCED VOLUME v_{n-1} / v_n ON ∂C_{n-1}

FROM THE NEXT SEQUENCE, $v_{n-2} = v_{n-1} \left(\frac{w_{n-2}}{v_{n-2}} \right) = \left(\frac{w_{n-1}}{v_{n-1}} \right) \left(\frac{w_{n-2}}{v_{n-2}} v_{n-1} \right)$

SO THE INDUCED VOLUME ON ∂C_{n-2} IS $\frac{v_{n-1}}{v_{n-2}} w_{n-2}$. CONTINUING IN THIS

WAY WE GET $v_1 = \frac{1}{T_1} w_2 v_0 = (\text{INDUCED VOL ON } \partial C_2) \cdot \left(\frac{v_2 T_2 \dots v_0}{T_1 T_3 \dots} \right)$

AND THE FINAL INDUCED VOLUME ON C_0 IS $\frac{v_2 T_2 \dots v_0}{T_1 T_3 \dots} v_0$. THE

TORSION IS THEN $\frac{v_0}{\text{FINAL VOL}} = T_1 T_2^{-1} T_3 T_4^{-1} \dots$ AS CLAIMED.

C. THE DUAL SEQUENCE $0 \rightarrow C_0^* \xrightarrow{d_1} C_1^* \xrightarrow{d_2} \dots \xrightarrow{d_{n-1}} C_{n-1}^* \rightarrow 0$ HAS

DUAL VOLUMES $\{v_i^*\}$ SO WE CAN COMPUTE $\tau^* = \tau(C^*)$. LET $\tau(C_i) = \tau$.

PROP 17 $\tau^* = (-1)^{\sum_{i=1}^{n-1} \beta_i \beta_{i+1}} \tau^{(-1)^{n-1}}$ WHERE $\beta_1 = \dim C_0, \beta_2 = \dim \partial C_2,$

$\dots, \beta_{n-1} = \dim \partial C_{n-1}$ AND $\beta_n = \dim C_n$.

PROOF: CHOOSE VOLUMES w_i ON ∂C_i AS ABOVE. WE HAVE $\int E \int$ 'S

$C_0^* \rightarrow C_1^* \xrightarrow{d_1} (\partial C_2)^*$

$(\partial C_2)^* \rightarrow C_2^* \xrightarrow{d_2} (\partial C_3)^*$

$(\partial C_{n-1})^* \rightarrow C_{n-1}^* \rightarrow C_n^*$

$$w_i^* \text{ on } (2C_i)^* \text{ so we get number } \sigma_{n+1}^* = \frac{w_1^* w_2^*}{v_1^*} = (-1)^{\beta_1 \beta_2} \frac{(v_1 v_2)^*}{v_1^*} = (-1)^{\beta_1 \beta_2} \frac{v_1^* v_2^*}{v_1^*} = (-1)^{\beta_1 \beta_2} v_2^*$$

$$\text{Likewise, } \sigma_{n-2}^* = \frac{w_2^* w_3^*}{v_2^*} = (-1)^{\beta_2 \beta_3} \frac{(w_2 w_3)^*}{v_2^*} = (-1)^{\beta_2 \beta_3} \frac{(v_2 v_3)^*}{v_2^*} = (-1)^{\beta_2 \beta_3} v_3^*$$

$$\sigma_i^* = (-1)^{\beta_{i-1} \beta_i} v_i^* \quad \text{Hence } \sigma_i^* = (-1)^{\sum_{j=1}^{i-1} \beta_j \beta_{j+1}} v_i^* \quad (-1)^{n-1}$$

$$\text{Hence } \sigma^* = (-1)^{\sum_{i=1}^n \beta_i \beta_{i+1}} \sigma$$

REMARK: THE SIGN IS ANNOYING, BUT WE WILL ONLY BE COMPARING DIMENSIONS FOR A SPECIAL KIND OF COMPLEX, PROMPTING THE FOLLOWING:

DEFINITION: a) A SEQUENCE OF INTEGERS $\beta_0, \beta_1, \dots, \beta_n$ IS BALANCED IF $\beta_i = \beta_{n-i}$ ALL i
 b) AN ACYCLE SEQUENCE C_i IS BALANCED IF THE DIMENSIONS d_i OF C_i ARE BALANCED.

PROP 15 LET $C_n = (0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_0 \rightarrow 0)$ BE A BALANCED SEQUENCE OF VOLUMED VECTOR SPACES, WHERE n IS ODD. THEN $\sigma(C_n^*) = \sigma(C_n)$, I.E. $\sigma^* = \sigma$.

PROOF: BY PROP. 14, WE NEED ONLY SHOW $\sum_{i=1}^{n-1} \beta_i \beta_{i+1}$ IS EVEN.

Now $\beta_0 = d_0 = d_n = \beta_n$
 Also $\beta_n + \beta_{n-1} = d_{n-1} = d_1 = \beta_2 + \beta_1$, so $\beta_2 = \beta_{n-1}$
 BY INDUCTION, $\beta_i = \beta_{n-1-i}$. THUS, THE β_i 'S ARE ALSO BALANCED.

BUT THEN $\beta_1 \beta_2 = \beta_{n-1} \beta_n$
 $\beta_2 \beta_3 = \beta_{n-2} \beta_{n-1}$, ETC., I.E. THE SEQUENCE $\{\beta_i \beta_{i+1}\}$ IS BALANCED.

BUT $\sum_{i=1}^{n-1} \beta_i \beta_{i+1}$ HAS AN EVEN NUMBER OF TERMS, SINCE n IS ODD, SO THE SUM IS EVEN, AS DESIRED.

D THE USEFULNESS OF THE VOLUME FORMULA COMES FROM THE FACT THAT A CELL COMPLEX HAS UP TO \dim , A CANONICAL VOLUME ON EACH CHAIN GROUP, NAMELY THAT GIVEN BY CHOOSING THE CELLS AS

BASIS. TO BE SPECIFIC LET π BE A GROUP ACTING FREELY ON A CELL COMPLEX \tilde{K} , I.E. π PERMUTES THE CELLS FREELY, AND ASSUME THAT $K = \tilde{K}/\pi$ IS A FINITE COMPLEX. THEN $C_*(K)$ CONSISTS OF FREE $\mathbb{Z}\pi$ -MODULES WITH BASIS GIVEN BY ORDERING AND ORIENTING THE CELLS OF K AND CHOOSING LIFTS OF THESE CELLS TO \tilde{K} .

NOW LET M BE AN $F\pi$ MODULE, F SOME FIELD, AND LET $\{m_i\}$ ($i=1, 2, \dots, d = \dim M$) A BASIS OF M OVER F . THEN WE GET CHAIN COMPLEXES $M^* \otimes C_*(K)$, $\text{Hom}_\pi(C_*(K), M)$, WITH BASES AND HENCE VOLUMES, COMING FROM THOSE OF \tilde{K} AND M AS FOLLOWS: LET m_i^* ($i=1, \dots, d$) BE THE DUAL BASIS OF M^* . THEN OUR PREFERRED BASIS IN $M^* \otimes C_i(K)$ WILL BE

$$m_1^* \otimes \tilde{\sigma}_1, m_2^* \otimes \tilde{\sigma}_2, m_3^* \otimes \tilde{\sigma}_3, \dots, m_d^* \otimes \tilde{\sigma}_d, m_1^* \otimes \tilde{\sigma}_2, m_2^* \otimes \tilde{\sigma}_1, \dots, m_1^* \otimes \tilde{\sigma}_d, \dots, m_d^* \otimes \tilde{\sigma}_1$$

WHERE $\tilde{\sigma}_i$ IS THE NUMBER OF i -CELLS IN K , $= \text{RANK } C_i$. SINCE $\text{Hom}(C_i, M)$ IS CANONICALLY DUAL TO $M^* \otimes C_i$, WE GIVE IT THE DUAL BASIS AND HENCE ALSO THE DUAL VOLUME.

AT THIS POINT, THE VOLUMES DEPEND ON SEVERAL CHOICES:

- 1) THE CHOICE OF LIFTS OF THE CELLS TO \tilde{K} .
- 2) THE ORIENTATION AND ORDERING OF THE CELLS OF K .
- 3) THE CHOICE OF BASIS $\{m_i\}$ OF M .

TO DEAL WITH PROBLEM 1), CONSIDER THE CHANGE IN THE VOLUME OCCURRING IF WE CHANGE THE LIFT $\tilde{\sigma}_i$ TO $x \tilde{\sigma}_i$, $x \in \pi$. THEN THE ORIGINAL BASIS CHANGES ONLY AT $m_r^* \otimes \tilde{\sigma}_i$ ($r=1, \dots, d$), CHANGING TO $m_r^* \otimes x \tilde{\sigma}_i = m_r^* x \otimes \tilde{\sigma}_i$. HENCE THE RATIO OF THE NEW VOLUME TO THE OLD ONE IS THE SAME AS THE CHANGE CAUSED BY THE ACTION OF x ON M^* . ALTHOUGH IT IS POSSIBLE

TO KEEP TRACK OF THESE CHANGES, OUR NEEDS ARE SIMPLER, AND WE 33.
MAKE THE FOLLOWING

DEFINITION: AN FTM MODULE M IS UNIMODULAR IF EVERY $\chi \in \Gamma$ PRESERVES VOLUME ON M , I.E., IF χ ACTS TRIVIAALLY ON $\Lambda^d M$.
(NOTE THAT THIS DOES NOT DEPEND ON A CHOICE OF VOLUME ON M .)

EXAMPLE 1: IF $\rho: \Gamma \rightarrow G$, G A (LINEAR) LIE GROUP, THEN Γ ACTS UNIMODULARLY ON THE LIE ALGEBRA. THIS IS BECAUSE (FOR LINEAR GROUPS) IT ACTS BY CONJUGATION, AND CONJUGATION IS ALWAYS VOLUME PRESERVING.

EXAMPLE 2: IF M IS UNIMODULAR, SO IS M^* .

EXAMPLE 3: LET M HAVE A Γ -INVARIANT PAIRING $M \otimes M \rightarrow F$. THEN M IS NEARLY UNIMODULAR, VARYING AT MOST BY A SIGN CHANGE. THIS IS BECAUSE THERE IS AN INDUCED ^{INVARIANT} PAIRING $\Lambda^d M \otimes \Lambda^d M \cong F \otimes F$ TO F , I.E. $\langle \chi v, \chi v \rangle = \langle v, v \rangle$, SO χv MUST BE $\pm v$.

EXAMPLE 4: IF THE PAIRING ON M IS ANTISYMMETRIC, I.E. SYMPLECTIC, THEN THERE IS NO SIGN INDETERMINACY. THIS IS BECAUSE THERE IS A CANONICAL VOLUME ON M^* GIVEN BY $\Theta^{\Lambda^2(\text{dual } M)}$ WHERE $\Theta = \sum_{i,j} \alpha_i^* \wedge \alpha_j^*$ IS THE SYMPLECTIC FORM ON $\Lambda^2 M^*$.

~~EX~~ NOTE THAT UNIMODULARITY IS THE SAME AS SAYING THAT THE ACTION OF Γ ON M IS IN $SL(M)$.

HENCEFORTH, ALL OUR COEFFICIENT MODULES WILL BE UNIMODULAR ~~AND SO THE INDETERMINACY DUE TO LIFTING~~ ^{AND SO THE INDETERMINACY DUE TO LIFTING} ~~CELLS DIFFERENTLY DISAPPEARS.~~

E. WE ARE LEFT WITH THE INDETERMINACY OF VOLUMES DUE TO:

- 2) THE ORIENTATION AND ORDERING OF THE CELLS OF K
- 3) THE CHOICE OF BASIS $\{m_i\}$ OF M .

NOTE THAT 2) CAN CAUSE ONLY A CHANGE OF SIGN. MOST WRITERS ON TORSION (E.G., MILNOR, DE RHAM) TYPICALLY IGNORE THIS ISSUE AND LEAVE THE TORSION DEFINED ONLY UP TO SIGN, BUT WE NEED TO RESOLVE THIS INDETERMINACY SUFFICIENTLY WELL TO SET

A WELL DEFINED TORSION. ~~LET A BE FREE ABELIAN OF RANK d~~

DEFINITION - LET A BE FREE ABELIAN OF RANK d; AN ORIENTATION OF A IS A CHOICE (2 POSSIBILITIES!) OF GENERATOR OF $\Lambda^d A$.

THIS IS EQUIVALENT TO A CHOICE OF ORIENTATION, IN THE USUAL SENSE, OF $A \otimes \mathbb{R}$. NOTE THAT A CHOICE CAN BE REPRESENTED BY $\alpha_1 \wedge \dots \wedge \alpha_d$, $\{\alpha_i\}$ A BASIS OF A.

GIVEN A ~~VOLUME~~ VOLUME θ ON M, SAY $\theta = \alpha_1 \wedge \dots \wedge \alpha_m$, AND AN ORIENTATION t ON $C_i(K, \mathbb{Z})$, THERE IS A WELL DEFINED VOLUME ON $M^* \otimes C_i(\mathbb{R})$ AS FOLLOWS: CHOOSE COVERED, ORIENTED CELLS σ_i IN K SUCH THAT $t = \sigma_1 \wedge \dots \wedge \sigma_{d_i}$ IN $C_i(K, \mathbb{Z})$. LIFT THESE CELLS ARBITRARILY TO \tilde{K} AND USE THE BASES $\{\alpha_r\}$ AND $\{\tilde{\sigma}_i\}$ TO CONSTRUCT OUR PREFERRED BASIS FOR $M^* \otimes C_i(\mathbb{R})$. THIS VOLUME IS WELL DEFINED INDEPENDENT OF CHOICES, DEPENDING ONLY ON θ AND t . WE WRITE IT AS $v_i(\theta^*, t)$, AND $v_i(t, \theta)$ FOR THE DUAL VOLUME ON $\text{Hom}(C_i, M)$.

- PROD 16**
- a) IF $f \in F$, $v_i(f\theta^*, t) = f^{-d_i} v_i(\theta^*, t)$ WHERE $d_i = \#i\text{-CELLS IN } K$
 - b) $v_i(t, f\theta) = f^{d_i} v_i(t, \theta)$
 - c) $v_i(\theta^*, -t) = (-1)^d v_i(\theta^*, t)$ WHERE $d = \dim M$

PROOF: CHANGING θ TO $f\theta$ CHANGES θ^* TO $f^{-1}\theta^*$ AND SO CHANGES THE VOLUME OF $M^* \otimes C_i \cong (M^*)^{d_i}$ BY f^{-d_i} . THE PROOF OF C) IS SIMILAR, AND b) FOLLOWS BY THE DUALITY OF VOLUMES IN \otimes AND Hom .

PROD 17 IF C_+ IS ACYCLIC, THEN THE TORSION $\tau(C_+)$ IS INDEPENDENT OF θ .

PROOF: CHANGING θ TO $f\theta$ CHANGES $v_i = v_i(\theta^*, t)$ BY f^{-d_i} , SO THE TORSION CHANGES BY $f^{-d_0} f^{d_1} f^{-d_2} \dots = f^{-\chi(C_+)} = f^0 = 1$, QED

F. THE INDETERMINACY: THE TORSION IS NOW REDUCED TO A SIGN DEPENDING ON CHOICES OF ORIENTATION OF $C_i(K, \mathbb{Z})$, OR EQUIVALENTLY, $C_i(K, \mathbb{R})$. NOTE THAT IF $\dim M$ IS EVEN, ORIENTATION CHANGES DO NOT AFFECT τ , BUT IF $\dim M$ IS ODD, τ MAY BE AFFECTED.

To make the dependence clearer, we make the

DEFINITION: An orientation of $C_*(K, Z)$, or equivalently $C_*(K, \mathbb{R})$, is an orientation of $\bigoplus C_i = C_0 \oplus C_1 \oplus \dots \oplus C_n$ (the order is important here!)

Note that this orientation can be represented by t_0, t_1, \dots, t_n , $n = \dim K$, t_i an orientation of C_i . Now given a vol θ on M , we get volumes $v_i(\theta, t_i)$ on $M \otimes C_i(\mathbb{R})$ and hence can compute torsion.

Prop 18 The torsion of (C_*, θ, t_i) depends only on t_i ; changing t_i to $-t_i$ changes the torsion τ to $(-1)^d \tau$ ($d = \dim M$)

- Proof: Changing t_i to $\epsilon_i t_i$ ($\epsilon_i = \pm 1$) causes the following changes
- a) $v_i \rightarrow (\epsilon_i)^d v_i$ by Prop 16 c) ; hence:
 - b) $\tau \rightarrow (\prod \epsilon_i)^d \tau$; also:
 - c) $t \rightarrow (\prod \epsilon_i) t$

Hence the sign change in τ is the d^{th} power of that in t .

Determining the sign of the torsion has ^{now} been reduced to a choice of orientation of $C_*(K, Z)$. We will see later how an oriented manifold M determines a canonical orientation of $C_*(M, Z)$ for any cell structure of M , thus making the sign of the torsion well defined (there is a restriction: $\dim M$ not divisible by 4).

G. First an example: ~~in a 3-dimensional lens space~~ $L(m, n)$. Recall from IV-E that we chose x, y such that $\begin{vmatrix} m & n \\ x & y \end{vmatrix} = 1$ and got the chain complex

$$0 \rightarrow \mathbb{C} \xrightarrow{\text{times } A^{-1}} \mathbb{C} \xrightarrow{\text{times } A^{-1}} \mathbb{C} \rightarrow 0, \text{ where } A \text{ can}$$

be any root of unity $e^{2\pi i/n}$, and the \mathbb{C} 's representing a simple cell in each dimension, the cell itself represented by ± 1 in \mathbb{C} .

Now complex multiplication on \mathbb{C} is not unimodular; however, if we think of \mathbb{C} as \mathbb{R}^2 with the usual area form, then mult by $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is unimodular. The area change on \mathbb{R}^2 given by multiplication by a complex z is just $|z|^2$. Thus, using the cellular basis given by $\pm 1 \in \mathbb{C}$ for each C_i , we can compute the torsion. Also, since $\dim \mathbb{R}^2$ is even, the torsion is independent of cell orientation, i.e. choice of signs ± 1 , so we can use $+1$; these give the volume basis over \mathbb{C} (i.e., $1, i$ is the volume basis over \mathbb{R}^2). We get

$$C_3 \xrightarrow{A^{-1}} C_2 \rightarrow \alpha_2 = 0 : \tau_2 = \frac{v_3 w_2}{v_2 w_1} = \frac{-v_3}{v_2} = \frac{|A^{-1}|^2 v_2}{v_2} = |A^{-1}|^2$$

$$\alpha_2 = 0 \rightarrow C_1 \xrightarrow{A^{-1}} C_0 : \tau_1 = \frac{w_2 v_0}{v_1} = \frac{v_0}{v_1} = \frac{v_0}{|A^{-1}|^2 v_0} = |A^{-1}|^{-2}$$

$$\tau = \tau_1 \tau_2^{-1} = |(A^{-1})(A^{-1})|^{-2}$$

H. We now examine the problem of orienting $C_*(K, \mathbb{Z})$ when K is an ~~oriented~~ oriented manifold. Note that we are not requiring $G(K, \mathbb{Z})$ to be acyclic; for computing torsion, acyclicity is only required in the complex $M^* \otimes \mathbb{C}(\tilde{K})$. Also, we reduce $C_*(K, \mathbb{Z})$ by the equivalent setting $C_*(K, \mathbb{R})$.

We begin by showing how an orientation on $\bigoplus H_i(K, \mathbb{R}) = H_0 \oplus H_1 \oplus \dots \oplus H_n$ ($n = \dim K$) induces a natural orientation on $G(K, \mathbb{R})$. This fixes the relationship between orientations of different complexes representing the same underlying space by relating them all to the invariant object $\bigoplus H_i$. This relationship works for any complex K , not just manifolds.

LET THEN h BE AN ORIENTATION OF $H_* = \bigoplus H_i$. h CAN ... 37.
 BE REPRESENTED BY ORIENTATIONS h_0, h_1, \dots, h_n ON H_i 'S, AND h
 IS WELL DEFINED UP TO AN EVEN NUMBER OF SIGN CHANGES OF THE h_i 'S.

TO SEE HOW h INDUCES AN ORIENTATION ON C_n , LK ON $\bigoplus C_i$,
 NOTE FIRST THAT IF $\alpha \rightarrow A \rightarrow B \rightarrow C \rightarrow \alpha$ IS AN SES OF \mathbb{R} -VECTOR SPACES,
 ORIENTATIONS ON ANY TWO DETERMINES A NATURAL ORIENTATION ON THE
 THIRD. THIS IS DONE IN EXACTLY THE SAME WAY AS WITH VOLUMES.
 BUT NOW WE HAVE THESE SES'S (LEAVING OUT INITIAL/FINAL 0'S):

$$\begin{aligned} \mathbb{Z}_n = H_n &\rightarrow C_n \rightarrow \mathbb{Z}_{n-1} = \partial C_n \\ \mathbb{B}_{n-1} &\rightarrow \mathbb{Z}_{n-1} \rightarrow H_{n-1} \\ \mathbb{Z}_{n-1} &\rightarrow C_{n-1} \rightarrow \mathbb{B}_{n-2} \\ &\vdots \\ \mathbb{B}_1 &\rightarrow \mathbb{Z}_1 \rightarrow H_1 \\ \mathbb{Z}_1 &\rightarrow C_1 \rightarrow \mathbb{B}_0 \\ \mathbb{B}_0 &\rightarrow C_0 \rightarrow H_0 \end{aligned}$$

CHOOSE ARBITRARY ORIENTATIONS b_i ON THE \mathbb{B}_i 'S. STARTING WITH
 THE FIRST SEQUENCE, h_n AND b_{n-1} INDUCE AN ORIENTATION γ_n ON C_n .
 ALSO, b_{n-1} AND h_{n-1} INDUCE ORIENTATION γ_{n-1} ON \mathbb{Z}_{n-1} . AGAIN,
 γ_{n-1} AND b_{n-2} INDUCE γ_{n-1} ON C_{n-1} , ETC. IN THIS WAY WE GET
 ORIENTATIONS ON ALL THE C_i , DEPENDING ON THE b_i . BUT NOTE THE
 EFFECT OF CHANGING b_i TO $-b_i$: FROM THE SEQUENCES.

$$\begin{aligned} H_{i+1} &\rightarrow C_{i+1} \rightarrow \mathbb{B}_i \\ \mathbb{B}_i &\rightarrow \mathbb{Z}_i \rightarrow H_i \\ \mathbb{Z}_i &\rightarrow C_i \rightarrow \mathbb{B}_{i-1} \end{aligned}$$

WE SEE THAT CHANGING b_i ALSO CHANGES γ_{i+1} AND γ_i , AND
 THE LATTER CAUSES A CHANGE IN γ_i . THUS, TWO SIGN CHANGES

HAVE BEEN MADE, AND SO THE ORIENTATION ON $\oplus C_i$ IS UNCHANGED.

IT REMAINS NOW TO FIND A NATURAL ORIENTATION ON $\oplus H_i$.

PROP 19 LET M BE A CLOSED ORIENTED MANIFOLD OF DIMENSION n NOT DIVISIBLE BY 4. THEN THERE EXISTS A NATURAL ORIENTATION ON $\oplus H_i$.

THE PROOF IS JUST A CONSTRUCTION. LET H_0, H_1, \dots, H_k BE THE GROUPS BELOW THE MIDDLE DIMENSION, I.E. LET $k = \lfloor \frac{n-1}{2} \rfloor$. GIVE THEM ARBITRARY ORIENTATIONS. THEN THE GROUPS $H_n, H_{n-1}, \dots, H_{n-k}$, BEING CANONICALLY DUAL TO THE FIRST SEQUENCE, ACQUIRE THE DUAL ORIENTATIONS. A CHANGE IN THE ORIENTATION OF H_i ALSO CAUSES A SIGN CHANGE IN THE DUAL H_{n-i} , SO NO RESULTING CHANGE IN THE TOTAL ORIENTATION. THE ONLY REMAINING PROBLEM IS $H_{n/2}$ WHEN n IS EVEN. BY HYPOTHESIS, $n/2$ IS ODD AND SO THE SELF-DUALITY ON $H_{n/2}$ IS SYMPLECTIC, AND HENCE IT HAS A NATURAL ORIENTATION ALSO. THIS COMPLETES THE CONSTRUCTION.

WE HAVE NOW SHOWN HOW TO GIVE AN ORIENTATION TO $G_*(K, Z)$ FOR ANY COMPLEX REPRESENTING THE SAME UNDERLYING MANIFOLD AND HENCE HOW TO GIVE THE TORSION A WELL DEFINED SIGN.

NOTE: THE ABOVE PROCEDURE OF SETTING A NATURAL ORIENTATION ON $\oplus H_i$ AND TRANSFERRING IT TO AN ORIENTATION OF $\oplus C_i$ CAN ALSO BE CARRIED OUT FOR VOLUMES. THAT IS:

NOTE: LOCAL ORIENTATIONS \rightarrow

a) VOLUMES ON H_i DETERMINE VOLUMES ON THE C_i 'S

~~THE CLASS OF VOLUMES ON H_i IS IN ONE-TO-ONE CORRESPONDENCE WITH THE CLASS OF VOLUMES ON C_i . THE ORIENTATION ON H_i DETERMINES THE ORIENTATION ON C_i .~~

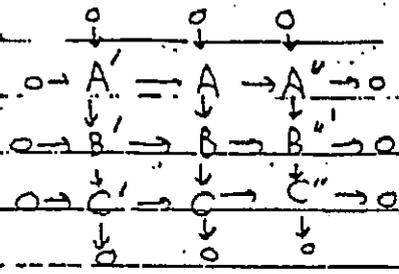
NOTE: LOCAL SELF-DUAL ORIENTATIONS \rightarrow

b) IF K IS AN EVEN DIMENSIONAL MANIFOLD, THEN PONCARE GIVES A CLASS OF VOLUMES ON $\{H_i\}$ WHOSE CORRESPONDING VOLUMES ON $\{C_i\}$ ALL HAVE THE SAME TORSION (WELL DEFINED ONLY UP TO SIGN FOR $4k$ -MANIFOLDS).

HENCE FOR EVEN DIMENSIONAL MANIFOLDS, TORSION CAN BE DEFINED EVEN WHEN C_* IS NOT ACYCLIC, AND IT IS A \mathbb{P} -INVARIANT OF THE MANIFOLD:

J. HERE ARE TWO MORE MEANS OF MANIPULATING TORSION,

PROP 20 - LET



BE A DIAGRAM OF VECTOR SPACES WITH EXACT ROWS AND COLUMNS.

LET VOLUMES $\theta_{A'}, \theta_{A''}, \theta_{C'}, \theta_{C''}$ BE GIVEN IN THE CORNERS,

WHICH DETERMINE VOLUMES $\theta_A = \theta_{A'} \theta_{A''}, \theta_B = \theta_{A'} \theta_{C'}, \theta_C = \theta_{C'} \theta_{C''}$ AND

$\theta_B = \theta_{A'} \theta_{C''}$ ON THE CORRESPONDING VECTOR SPACES. THEN THE

TWO VOLUMES $\theta_B, \theta_{B''}$ ON B SATISFY $\theta_B \theta_{B''} = (-1)^{\text{rank } A} \theta_{A'} \theta_{C''}$

PROOF: CHOOSE IDENTIFICATIONS $A \cong A' \oplus A'', C \cong C' \oplus C''$

$B' \cong A' \oplus C', B'' \cong A'' \oplus C''$ AND $B = A' \oplus A'' \oplus C' \oplus C''$ BEING

THE MAP OF THE DIAGRAM. THEN $\theta_B = \theta_{A'} \theta_{A''}$ IS JUST $\theta_{A'} \Delta \theta_{A''}$

AND LIKEWISE FOR THE OTHER VOLUMES. THEN THE TWO INDUCED

VOLUMES ON B ARE JUST $\theta_{A'} \Delta \theta_{A''} \Delta \theta_{C'} \Delta \theta_{C''}$ AND $\theta_{A'} \Delta \theta_{C'} \Delta \theta_{A''} \Delta \theta_{C''}$,

WHICH CLEARLY DIFFER BY THE SIGN CLAIMED IN THE PROPOSITION.

PROP 21

LET $0 \rightarrow C'_* \rightarrow C_* \rightarrow C''_* \rightarrow 0$ BE AN EXACT SEQUENCE

OF VOLUMED COMPLEXES OF DIM n (I.E. DEGREE OF HIGHEST NON ZERO SPACE = n)

SUCH THAT $\partial_i = \partial'_i \partial''_i$ FOR ALL i , AND LET $\rho'_i = \dim \partial C'_i$

($i \in \{1, \dots, n\}$), $\rho''_i = \dim \partial C''_i$. IF ANY TWO OF THE COMPLEXES ARE

ACYCLIC, THEN SO IS THE THIRD AND THE TORSIONS τ'_i, τ''_i ARE ALL

DEFINED AND SATISFY

$$\tau = (-1)^{\sum_{i=1}^{n-1} \rho'_i \rho''_i} \tau' \tau''$$

PROOF: THE ACYCLICITY OF THE THIRD CHAIN COMPLEX FOLLOWS

FROM THE LONG EXACT SEQUENCE OF Homology $C'_* \rightarrow C_* \rightarrow C''_*$. NOW

FOR EACH i WE HAVE A DIAGRAM OF THE TYPE IN PROP 20:

$$\begin{array}{ccccccc}
 \partial C_{i+1} & \rightarrow & \partial C_{i+1} & \rightarrow & \partial C_{i+1} & \rightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 C_i & \rightarrow & C_i & \rightarrow & C_i & \rightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \partial C_i & \rightarrow & \partial C_i & \rightarrow & \partial C_i & \rightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & &
 \end{array}$$

To compute the torsions, choose volumes w_i' in $\partial C_i'$ and w_i'' in $\partial C_i''$ and use the volumes $w_i = w_i' w_i''$ for ∂C_i .

We now compute the numbers $\tau_i', \tau_i, \tau_i''$ by definition.

$\tau_i' = \frac{w_{i+1}' w_i'}{v_i'}$ i.e. $\tau_i' v_i' = w_{i+1}' w_i'$ and similarly

~~.....~~ $\tau_i v_i = w_{i+1} w_i, \tau_i'' v_i'' = w_{i+1}'' w_i''$

We have thus a diagram of volumes

w_{i+1}'	$w_{i+1} = w_{i+1}' w_{i+1}''$	w_{i+1}''
$\tau_i' v_i' = w_{i+1}' w_i'$		$\tau_i'' v_i'' = w_{i+1}'' w_i''$
w_i'	$w_i = w_i' w_i''$	w_i''

Hence by Proposition 20, $\tau_i' \tau_i'' v_i' v_i'' = (-1)^{\beta_i' \beta_i''} w_{i+1}' w_i''$
 $= (-1)^{\beta_i' \beta_i''} \tau_i v_i = (-1)^{\beta_i' \beta_i''} \tau_i v_i' v_i''$, i.e.
 $\tau_i = (-1)^{\beta_i' \beta_i''} \tau_i' \tau_i''$. But $\tau = \tau_1 \tau_2^{-1} \tau_3$
 and likewise for τ', τ'' , so $\tau = (-1)^{\beta_i' \beta_i''} \tau' \tau''$

VIII. VOLUMES FOR FREE GROUPS

Let Π be a free group, M a unimodular F -module, and let $\theta = \alpha_1 \alpha_2 \dots \alpha_m$ ($d = \dim M$) be a volume on M . We will construct

a natural volume on $M^* \otimes d\Pi$ which depends only on θ and an orientation t of the free abelian group $A = \Pi/\Pi'$. Let

x_1, x_2, \dots, x_m be a free basis of Π . This gives a basis $\{dx_i\}$ of $d\Pi$. In exactly the same way as we constructed

volumes on $M^* \otimes C_1(\tilde{K})$ in VII.E, we get a volume $\nu(\theta^*, t)$ on $M^* \otimes d\Pi$, and ν satisfies the same law as

in Prop 16. In fact using the given basis $\{x_i\}$ we can identify $d\Pi$ with $C_1(\tilde{K})$, where \tilde{K} is the universal cover

of the 1 -complex K which has one vertex and m edges $\{x_i\}$. In this way A becomes identified with $C_1(K/\mathbb{Z})$, and

then $\nu(\theta^*, t)$ is exactly the volume we constructed in VII.E. The main thing we want to see here is that the volume ν

does not depend on the choice of ~~the~~ free basis $\{x_i\}$. Now given a basis $X = (x_1, \dots, x_m)$ we get a

corresponding basis (a_1, \dots, a_m) of A and hence an orientation $t_x = a_1 \wedge \dots \wedge a_m$ of A , and so then a volume $\nu_x = \nu(\theta^*, t_x)$ on

$M^* \otimes d\Pi$. A volume basis for ν_x is given by $m_1^* \otimes dx_1, m_2^* \otimes dx_1, \dots, m_d^* \otimes dx_1, m_1^* \otimes dx_2, \dots, m_d^* \otimes dx_m$.

Therefore we define $\epsilon_x = \pm 1$ by $t = \epsilon_x t_x$, we have (by Prop 16) $\nu(\theta^*, t) = \epsilon_x \nu_x$. To see how $\nu(\theta^*, t)$ behaves

under a change of the free basis, we need only see how it changes under the three Nielsen moves:

- I) $y_i = x_j, y_j = x_i, \dots$ other basis elements unchanged
- II) $y_i = x_i^{-1}, \dots$ OITTO
- III) $y_i = x_i x_j, \dots$ OITTO

Since these moves generate all basis changes in Π .

Prob. 22

UNDER THE NIELSEN MOVES WE HAVE:

I) $v_y = (-1)^d v_x$, $E_y = -E_x$ so $v(\theta^t, t)$ IS UNCHANGED

II) $v_y = (-1)^d v_x$, $E_y = -E_x$, DITTO

III) $v_y = v_x$, $E_y = E_x$, DITTO

PROOF:

I) $dy_i = dx_j$, $dy_j = dx_i$, SO WE ARE SWITCHING

$\{ \mu_r^* \otimes dx_i \}$ AND $\{ \mu_r^* \otimes dx_j \}$, AND THE VOLUME CHANGES BY

$(-1)^d$. $v_y = (-1)^d v_x$. BUT NOTE THAT THE ORIENTATION t_x CHANGES

TO $-t_x$, SINCE THE BASIS CHANGE IN A IS $a_i \leftrightarrow a_j$. TRUE

$t_y = -t_x$ AND SO $E_y = -E_x$. HENCE $E_y^d v_y = (-E_x)^d (-1)^d v_x = E_x^d v_x$

IE, $v(\theta^t, t)$ IS INVARIANT UNDER TYPE I) MOVES.

II) $dy_1 = -x_1^{-1} dx_1$, SO THE VOLUME BASIS FOR v_y DIFFERS FROM

THAT OF v_x ONLY AT $\{ \mu_r^* \otimes dx_1 \}$, BECOMING $\{ \mu_r^* \otimes -x_1^{-1} dx_1 \} =$

$\{ -\mu_r^* x_1^{-1} \otimes dx_1 \}$. SINCE M IS UNIMODULAR, THE VOLUME CHANGE IS

JUST $(-1)^d$, I.E. $v_y = (-1)^d v_x$. ALSO, t_y IS AGAIN $-t_x$, SINCE

THE BASIS CHANGE IN A IS JUST $a_1 \rightarrow -a_1$; THIS PROVES PART II).

III) $dy_1 = dx_1 + x_1 dx_2$, SO THE ONLY BASIS CHANGE IS AT

$\{ \mu_r^* \otimes dx_1 \}$, BECOMING $\{ \mu_r^* \otimes (dx_1 + x_1 dx_2) \} = \{ \mu_r^* \otimes dx_1 + \mu_r^* x_1 \otimes dx_2 \}$.

BUT THE TERMS $\mu_r^* x_1 \otimes dx_2$ ARE LINEAR COMBINATIONS OF $\{ \mu_r^* \otimes dx_2 \}$,

SO WHEN WE WEAVE ALL THE BASIS ELEMENTS TOGETHER WE GET THE SAME

VOLUME AS BEFORE. THE NEW ORIENTATION IS $t_y = (a_1, a_2, \dots, a_n) = t_x$

SO $E_y = E_x$. THIS COMPLETES THE PROOF.

USING THE DUAL VOLUME GIVES US ALSO NATURAL VOLUMES $v(t, \theta)$

ON $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, M) \cong \text{XHom}(\mathbb{Z}^n, M) \cong \mathbb{Z}^n(\mathbb{Z}^n, M)$.

LET G BE A LIE GROUP AND L ITS LIE ALGEBRA AS BEFORE;
 AND R_Π BE THE REP. SPACE OF THE FREE GROUP Π . BY
 A VOLUME ON R_Π WE MEAN A TOP DEGREE NOWHERE ZERO
 FORM ON R_Π . SINCE TR_Π AT p IS $\cong Z^1(\Pi, L_p)$,
 EVERY VOLUME θ ON L AND ORIENTATION t OF $A = \Pi/\Pi'$ DETERMINES
 A UNIQUE SUCH FORM ON R_Π , AND IT SATISFIES THE LAWS OF PROP. 16.
 WE CAN ALSO USE THIS VOLUME TO GET A VOLUME ON \hat{R}_Π AT
 THE IRREDUCIBLE REPS ρ AS FOLLOWS. THE COBOUNDARY MAP
 $L \xrightarrow{\delta} Z^1(\Pi, L_p)$ IS 1-1 AT ρ , SINCE $H^0(\Pi, L_p) = Z^0 = 0$,
 WE HAVE AN EXACT SEQUENCE:

$$E_1) \quad 0 \rightarrow L \xrightarrow{\theta} Z^1(\Pi, L_p) \xrightarrow{T(t, \theta)} H^1(\Pi, L_p) \rightarrow 0, \text{ WHERE WE}$$

HAVE INDICATED EXISTING VOLUMES BELOW THE 1ST TWO GROUPS.

HENCE WE GET A NATURAL VOLUME $\hat{v}(t, \theta) = v(t, \theta) / \theta$ ON
 $H^1(\Pi, L_p) = T\hat{R}_\Pi$ (AT EVERY ρ SUCH THAT $H^0(\Pi, L_p) = 0$).

THIS VOLUME IS INDEPENDENT OF FREE BASIS AND SATISFIES LAWS
 SIMILAR TO PROP. 16, I.E. ~~...~~

$$a) \hat{v}(t, \theta) = t^{n-1} \hat{v}(t, \theta) \quad (n = \text{RANK } \Pi)$$

$$b) \hat{v}(-t, \theta) = (-1)^d \hat{v}(t, \theta) \quad (d = \dim G = \dim L)$$

NEXT, LET Π BE A SURFACE GROUP WITH THE USUAL PRESENTATION
 $Z \xrightarrow{1-Z} \Pi \rightarrow \Pi \rightarrow 0$, WHERE Π IS FREE ON (GEOMETRICAL

GENERATORS) a_i, b_i AND $Z = \Pi[a_i, b_i]$. THE SURFACE ITSELF
 IS DENOTED BY K . WE SHOW HOW TO GET NATURAL VOLUMES
 ON R_Π^{**} AND \hat{R}_Π^{**} , I.E., ON EACH TANGENT SPACE $TR_\Pi(p), \hat{R}_\Pi(p)$,
 FOR IRREDUCIBLE ρ .

WE BEGIN BY NOTING THAT THERE IS A NATURAL ORIENTATION
 ON $A = \Pi/\Pi' \cong H_1(K, Z)$: THIS IS BECAUSE THE PAIRINGS ON H_1

IS SYMPLECTIC. So in this case we have a volume $\nu_{\pi}(\theta)$ on R_{π} which depends only on θ , independent of the choice of geometric basis a_i, b_i of K . [NOTE: our convention for order the a_i, b_i will be $a_1, a_2, \dots, a_g, b_1, b_2, \dots, b_g$ — NOT $a_1, b_1, a_2, b_2, \dots$]

Now, recall that for p irreducible, $H^0(\pi, L_p) = H^2(\pi, L_p) = 0$ (see Prop II of IV.D. and the following note). Using our standard complex $Z_{\pi} \xrightarrow{d_{\pi}} \pi^* d\pi \xrightarrow{\epsilon} Z_{\pi}$ to compute H^k , we get after hom'ing the above into L_p the following sequence

$$0 \leftarrow L_p \xrightarrow{EV_{\Sigma}} Z^1(\pi, L_p) \xrightarrow{\delta_0} L_p \leftarrow 0$$

where the boundary map $\delta_0 = EV_{\Sigma}$ is evaluation of $f \in Z^1$ on $d\Sigma$. The kernel of this map is then $Z^1(\pi, L_p) = TR_{\pi}(p)$, and EV_{Σ} is onto (since $H^2=0$), and δ_0 is ± 1 (since $H^0=0$). We can then write the following exact sequences:

$$E2) \quad 0 \rightarrow Z^1(\pi, L_p) \rightarrow Z^1(\pi, L_p) \xrightarrow{EV_{\Sigma}} L_p \rightarrow 0$$

$$\quad \quad \quad \nu_K(\theta) = \nu_{\pi}(\theta)/\theta \quad \quad \quad \nu_{\pi}(\theta) \quad \quad \quad \theta$$

$$E3) \quad 0 \rightarrow L_p \xrightarrow{\delta_0} Z^1(\pi, L_p) \rightarrow H^1(\pi, L_p) \rightarrow 0$$

$$\quad \quad \quad \theta \quad \quad \quad \nu_K(\theta) \quad \quad \quad \hat{\nu}_K(\theta) = \nu_K(\theta)/\theta$$

In E2) the existing volumes $\nu_{\pi}(\theta), \theta$ give us a natural volume $\nu_K(\theta)$ on $Z^1(\pi, L_p) = TR_{\pi}(p)$; then using it and θ again in E3) gives us $\hat{\nu}_K(\theta)$ on $H^1(\pi, L_p) = \hat{TR}_{\pi}(p)$. We repeat that these volumes do not depend on any indeterminate orientations.

X. THE TANGENTIAL COMPLEX

A. IN THIS SECTION WE CAN FINALLY DEFINE THE NUMBER t_p 45:

WHICH WE DESCRIBED IN SECTION I. RECALL THAT W IS

A 3-MANIFOLD WITH HEEGAARD DECOMPOSITION $A \cup_{\mathcal{Q}} \mathcal{Q}$ OF TWO GENUS g HANDLEBODIES A, \mathcal{Q} WITH COMMON BOUNDARY K .

LET $\pi_A, \pi_{\mathcal{Q}}$ BE THE (FREE) FUNDAMENTAL GROUPS OF A, \mathcal{Q} , AND π_K THAT OF K . THE SURJECTIVE HOMS $\pi_A \leftarrow \pi_K \rightarrow \pi_{\mathcal{Q}}$ INDUCE IMBEDDINGS $\hat{R}_A \hookrightarrow \hat{R}_K \hookrightarrow \hat{R}_{\mathcal{Q}}$ IF p IS AN (IRREDUCIBLE) REP IN $\hat{R}_A \cap \hat{R}_{\mathcal{Q}}$ - I.E. AN (CLOSED) REP OF $\pi_1(W)$ THEN WE HAVE A DIAGRAM OF THE TANGENT SPACES $TR_{\hat{R}_A}, TR_{\hat{R}_{\mathcal{Q}}} \subset TR_{\hat{R}_K}$:

$$TR_{\hat{R}_A} \oplus TR_{\hat{R}_{\mathcal{Q}}} \xrightarrow{\text{sum}} TR_{\hat{R}_K}$$

BECAUSE $\dim TR_K = d(2g-2) = \dim TR_A + \dim TR_{\mathcal{Q}}$, THIS SEQUENCE IS EXACT IFF $TR_A \pitchfork TR_{\mathcal{Q}}$. IN ANY CASE, THE VOLUMES

$\hat{v}_A = \hat{v}_A(\theta, t_A)$ AND $\hat{v}_{\mathcal{Q}} = \hat{v}_{\mathcal{Q}}(\theta, t_{\mathcal{Q}})$ IN THE DIRECT SUM, AND

$\hat{v}_K = \hat{v}_K(\theta)$ ON THE RIGHT, DETERMINE THE NUMBER $t_p = \frac{\hat{v}_A \wedge \hat{v}_{\mathcal{Q}}}{\hat{v}_K}$.

NOTE THAT $t_p = 0$ IFF $\hat{R}_A \not\pitchfork \hat{R}_{\mathcal{Q}}$ ARE NOT TRANSVERSE

AT p . ALSO, t_p DOES NOT DEPEND ON θ .

SO FAR, t_p IS DETERMINED ONLY UP TO A SIGN DETERMINED BY THE ORIENTATIONS $t_A, t_{\mathcal{Q}}$ OF $\pi_A/\pi_A' = H_1(A, \mathbb{Z}), \pi_{\mathcal{Q}}/\pi_{\mathcal{Q}}' = H_1(\mathcal{Q}, \mathbb{Z})$.

WE NOW GIVE A NATURAL ORIENTATION TO THE DIRECT SUM AND, EQUIVALENTLY SHOW HOW, FOR ANY CHOSEN ORIENTATION t_A , TO GIVE A NATURALLY "PAIRED" ORIENTATION ON $t_{\mathcal{Q}}$.

FROM THE MAVER-VIETORIS SEQUENCE OF THE HEEGAARD DECOMPOSITION WE EXTRACT THIS EXACT SEQUENCE:

$$0 \rightarrow H_2(W, \mathbb{R}) \rightarrow H_1(K, \mathbb{R}) \rightarrow H_1(A, \mathbb{R}) \oplus H_1(\mathcal{Q}, \mathbb{R}) \rightarrow H_1(W, \mathbb{R}) \rightarrow 0$$

THE FIRST AND LAST GROUPS ARE DUAL, SO CHOOSE AN ORIENTATION ON $H_2(W, \mathbb{R})$ AND THE DUAL ORIENTATION ON $H_1(W, \mathbb{R})$.

$H_1(K, \mathbb{R})$, BEING SYMPLECTIC, HAS A NATURAL ORIENTATION, SO WE GET AN INDUCED ORIENTATION ON THE QUOTIENT $H_1(K) / H_2(W)$, I.E., ON THE IMAGE OF $H_1(K)$ IN $H_1(A) \oplus H_1(Q)$. THIS IMAGE IS COMPLEMENTARY TO A SUBSPACE WHICH MAPS ISOMORPHICALLY ONTO $H_1(W, \mathbb{R})$ AND HENCE THIS SUBSPACE ACQUIRES AN ORIENTATION VIA THE ISOMORPHISM. THEN THE ORIENTATIONS ON THE TWO COMPLEMENTARY SUBSPACES GIVE AN ORIENTATION TO $H_1(A) \oplus H_1(Q)$. BUT THIS ORIENTATION IS INDEPENDENT OF CHOICES; FOR EXAMPLE, CHANGING THE ORIGINAL ORIENTATION ON $H_2(W)$ ALSO CAUSE CHANGES IN BOTH $H_1(W)$ AND $H_1(K) / H_2(W)$, SO LEAVES $H_1(A) \oplus H_1(Q)$ THE SAME.

HENCE FOR ANY ORIENTATION t_A OF $H_1(A, \mathbb{Z})$ THERE IS A CORRESPONDING ORIENTATION t_Q OF $H_1(Q, \mathbb{Z})$ SUCH THAT $t_A \oplus t_Q$ IS THE CANONICAL ORIENTATION OF $H_1(A) \oplus H_1(Q)$, AND REVERSING t_A ALSO REVERSES t_Q . PAIRING THE ORIENTATIONS IN THIS WAY ALSO GIVES A PAIRING OF VOLUMES $v_A(t_A, \theta)$, $v_Q(t_Q, \theta)$ ON TR_A, TR_Q AND $\hat{v}_A(t_A, \theta)$, $\hat{v}_Q(t_Q, \theta)$ ON \hat{TR}_A, \hat{TR}_Q , AND HENCE WELL DEFINED VOLUMES ON

$$TR_A \otimes TR_Q = \mathbb{Z}'(\pi_A, L_p) \oplus \mathbb{Z}'(\pi_Q, L_p) \text{ AND}$$

$$\hat{TR}_A \otimes \hat{TR}_Q = H^1(\pi_A, L_p) \oplus H^1(\pi_Q, L_p). \text{ IN PARTICULAR,}$$

THE "FRON" t_p BECOMES WELL DEFINED. [NOTE: IT CAN BE VERIFIED DIRECTLY THAT STRATIFICATION LEVELS t_p INVARIANT, THE FORMERS USE CHANGE BECAUSE OF AN ORIENTING BASIS.]

B. HAVING WORKED OUT WAY DOWN TO \hat{R} AND H^1 IN ORDER TO DEFINE t_p , WE NOW "BACK OUT" AND FIND A WAY TO COMPUTE t_p AT THE CHAIN LEVEL. THIS WILL PUT US IN A POSITION TO ACCOMPLISH OUR ~~PRIMARY~~ PRIMARY GOAL OF PROVING THAT t_p IS THE REIDEMEISTER TORSION OF W WITH L_p^* COEFFICIENTS.

FIRST, CONSIDER THE DIAGRAMS OF VECTOR SPACES AND CORRESPONDING VOLUMES!

$$\begin{array}{ccc}
 0 \rightarrow L \xrightarrow{\delta} Z(\pi_A, L) \rightarrow H^1(\pi_A, L) \rightarrow 0 & \theta & v_A & \hat{v}_A = v_A / \theta \\
 \downarrow (1) \downarrow & & \theta \wedge \theta & \hat{v}_A \wedge \hat{v}_\theta \\
 0 \rightarrow L \otimes \theta \xrightarrow{\delta \otimes 1} Z(\pi_A, L) \otimes \theta \rightarrow H^1(\pi_A, L) \otimes \theta \rightarrow 0 & & & \\
 \downarrow (1) \downarrow & & & \\
 0 \rightarrow L \xrightarrow{\delta} Z(\pi_\theta, L) \rightarrow H^1(\pi_\theta, L) \rightarrow 0 & \theta & v_\theta & \hat{v}_\theta = v_\theta / \theta
 \end{array}$$

THE ROWS AND COLUMNS ARE EXACT SO BY PROPOSITION 20 WE GET

$$\hat{v}_A \wedge \hat{v}_\theta = (-1)^{\dim L \cdot \dim H^1(\pi_A, L)} (v_A \wedge v_\theta) / (\theta \wedge \theta)$$

LEAVING SIGN: WE HAVE $\dim L = d$, $\dim H^1 = d(g-1)$, $\dim L \cdot \dim H^1 \equiv d(g-1) \pmod{2}$, SO:

$$S = (-1)^{d(g-1)}$$

$$\hat{v}_A \wedge \hat{v}_\theta = S (v_A \wedge v_\theta) / (\theta \wedge \theta)$$

NEXT CONSIDER

$$\begin{array}{ccc}
 0 \rightarrow L \xrightarrow{\delta} L \rightarrow 0 & \theta & \theta & 1 \\
 \downarrow (1) \downarrow & & & \\
 L \otimes \theta \xrightarrow{\delta \otimes 1} Z(\pi_A, L) \otimes \theta \rightarrow H^1(\pi_A, L) \otimes \theta \rightarrow 0 & \theta \wedge \theta & v_A \wedge v_\theta & S(\hat{v}_A \wedge \hat{v}_\theta) \\
 \downarrow \text{sum} \downarrow & & & \\
 L \xrightarrow{\delta} Z(\pi_A, L) \rightarrow H^1(\pi_A, L) \rightarrow 0 & \theta & v_K & \hat{v}_K = v_K / \theta
 \end{array}$$

THE ROWS AND COLUMNS ARE EXACT AND THE VOLUMES IN THE ROWS SATISFY

THE HYPOTHESES OF Prop 21: THE MIDDLE VOLUME IS THE "PRODUCT" OF THE ENDS.

HENCE IF τ', τ, τ'' ARE THE TORSIONS OF THE 3 COLUMNS, THAT PROPOSITION

GIVES US $\tau = \tau' \tau''$. NOW $\tau' = \hat{v}_K / S(\hat{v}_A \wedge \hat{v}_\theta) = S t_p^{-1}$ BY DEF.

OF t_p : ALSO, IT IS EASILY COMPUTED THAT $\tau'' = 1$. HENCE

WE GET THE TORSION OF THE MIDDLE COLUMN IS $S t_p^{-1}$.

WRITING THE MIDDLE COLUMN OUT, ALONG WITH SEQUENCE E2 FROM IX, WE GET:

$$\begin{aligned} 0 \rightarrow L \xrightarrow{(d_1, d_2)} Z'(\pi_A, L) \oplus Z'(\pi_B, L) \xrightarrow{\text{sum}} Z'(\pi_C, L) \rightarrow 0 & \text{torsion} = \tau_2 = s t_p^{-1} \\ 0 \rightarrow Z'(\pi_A, L) \rightarrow Z'(\pi_C, L) \xrightarrow{\text{ev}_C} L \rightarrow 0 & \text{torsion} = \tau_1 = 1 \end{aligned} \quad 48.$$

HENCE THE TORSION OF THE EXACT SEQUENCE

$$0 \rightarrow L_p \xrightarrow{(d_1, d_2)} Z'(\pi_A, L_p) \oplus Z'(\pi_B, L_p) \rightarrow Z'(\pi_C, L_p) \xrightarrow{\text{ev}_C} L_p \rightarrow 0$$

IS $\tau_1 \tau_2^{-1} = 1 \cdot (s t_p^{-1})^{-1} = s t_p$. THE LAST COMPLEX WE CALL THE TANGENTIAL COMPLEX, SINCE WE CAN IDENTIFY IT WITH

$$0 \rightarrow T(\text{CONNECTIONS}) \rightarrow TR_A \oplus TR_B \rightarrow TR_C \rightarrow \frac{TR_C}{TR_A} \rightarrow 0$$

IT WILL BE MORE CONVENIENT FOR US TO CALCULATE t_p FROM THE DUAL SEQUENCE, I.E. THE "COTANGENT COMPLEX"

$$0 \rightarrow L_p^* \xrightarrow{(d_1^*, d_2^*)} L_p^* \otimes \pi^* \rightarrow (L_p^* \otimes \pi_A^*) \oplus (L_p^* \otimes \pi_B^*) \xrightarrow{(d_1^*, d_2^*)} L_p^* \rightarrow 0$$

$$\begin{array}{c} \text{"} \\ L_p^* \otimes \pi^* \\ \text{"} \end{array} \quad \begin{array}{c} \text{"} \\ L_p^* \otimes (\pi_A^* \oplus \pi_B^*) \\ \text{"} \end{array}$$

WHERE $\pi = \pi_1(W^3)$ AND THE VERTICAL EQUALITIES HOLD BECAUSE ρ IS ACTUALLY A REPRESENTATION OF π . NOW SINCE THE COMPLEX IS 3-DIMENSIONAL AND BALANCED (THE MIDDLE GROUPS BOTH HAVE DIMENSION 2gd), BY PROP 15 WE HAVE (1) TORSION IS 0. $s t_p = (-1)^{d(g-1)} t_p$.

WE ARE NOW READY TO SHOW THAT t_p IS THE REINVENTED TORSION OF W^3 WITH L_p^* COEFFICIENTS.

XI. THE TANGENTIAL ISOMORPHISM

WE FIRST RECALL THE DEFINITION OF THE CHAIN COMPLEX $C_*(W)$ FROM III.D :

$$0 \rightarrow \sum \mathbb{Z} \langle (s_k - 1) d p_k \rangle \xrightarrow{\partial} \mathbb{Z} R \xrightarrow{d} \mathbb{Z} F \xrightarrow{\partial} \mathbb{Z} \Pi \rightarrow 0 \quad 49$$

WHERE $\Pi = \pi_1(W)$; $F =$ THE FREE GROUP $\pi_1 A = \pi_1$ OF THE HANDLES: $A =$ (0-HANDLE) \cup (g 1-HANDLES), GENERATED BY $\{a_i\}$ $i=1, \dots, g$; $R =$ THE FREE GROUP ON THE 2-HANDLES P_k ; $r: R \rightarrow F = \pi_1 A$ IS GIVEN BY $r(P_k) = \gamma_k(a_i)$, THE WORDS γ_k REPRESENTING THE ATTACHING MAPS. THE ATTACHING CURVES P_k IN THE SURFACE $K = \partial A$ ARE PART OF A GEOMETRIC BASIS γ_k, δ_k OF K (ORDERED WITH γ 'S FIRST, δ 'S LAST) AND THESE CURVES, AS ELEMENTS OF $\pi_1 A$, ARE GIVEN BY WORDS $\gamma_k(a_i)$ AND $\delta_k(a_i)$ RESPECTIVELY. THESE OBTAIN ALL THE ITEMS IN THE ABOVE COMPLEX.

WE RE-EXAMINE THE COMPLEX IN THE LIGHT OF THE ABOVE. NOTE THAT EVERYTHING IN THE COMPLEX IS TENSED WITH L_p^* ; TO KEEP THE NOTATION SIMILAR IN THE FOLLOWING WE ABBREVIATE Π TO THE FOLLOWING: $0 \rightarrow 1 \xrightarrow{d} \mathbb{Z} R \xrightarrow{d} \mathbb{Z} F \xrightarrow{\partial} \mathbb{Z} \Pi \rightarrow 0$. JUST KEEP IN MIND THAT EVERYTHING IS TENSED WITH L_p^* , AND THAT "1" REALLY MEANS L_p^* . LIKEWISE, WE WILL ABBREVIATE $C_*(W, L_p^*)$ TO $0 \rightarrow 1 \xrightarrow{\partial} \mathbb{Z} R \xrightarrow{d} \mathbb{Z} F \xrightarrow{\partial} 1 \rightarrow 0$.

WE NEED TO MAKE THE BOUNDARY MAPS OF THE COMPLEX MORE EXPLICIT. THE FIRST BOUNDARY MAP IS d_1^* UNDER $\mathbb{Z} = \Pi [p_i, b_i]$.

BECAUSE OF THE FACT THAT OUR COEFFICIENTS (I.E. ACTION) FACTORS THROUGH $\pi_1(W)$, THE γ_i 'S ACT TRIVIAALLY, I.E. $\gamma_i = 1$ IN π_1 . HENCE WE CALCULATE EASILY THAT $d_1^* = \sum_k (s_k - 1) d p_k$. ALSO, δ_k ACTS IN THE SAME WAY AS THE WORDS $\delta_k(a_i)$, SO WE CAN WRITE $d_1^* = \sum_k (s_k - 1) d p_k$. NOTE THAT NO $d \gamma$ 'S APPEAR IN d_1^* .

THE SECOND BOUNDARY MAP IS GIVEN BY THE PROJECTION MAPS OF Π TO $\pi_1 A$. TO MAKE THIS EXPLICIT, FIRST DECOMPOSE Π INTO THE DIRECT SUM OF THE FREE SUBMODULES GENERATED BY THE $d \gamma$ 'S AND

THE FIRST IS ~~ISOMORPHIC~~ ISOMORPHIC IN THE OBVIOUS WAY TO dR BY MAKING dp_i CORRESPOND TO dP_i . THE SECOND IS CLEARLY ISOMORPHIC SO TO $d\pi_Q$. HENCE WE CHANGE THE SYMBOLS dp_i TO dP_i AND WRITE $d\pi = dR \oplus d\pi_Q$. THE BOUNDARY MAP $dR \oplus d\pi_Q \rightarrow d\pi_A \oplus d\pi_Q$ IS THEN DETERMINED AS FOLLOWS:

ON dP_k WE GET FIRST THE PROJECTIONS OF $p_k \in \pi$ TO π_A, π_Q . BUT $p_k = 1$ IN π_Q AND $p_k = \gamma_k(q_i)$ IN π_A . HENCE $dP_k \rightarrow (dr_k, 0)$ IN $d\pi_A \oplus d\pi_Q$. ON THE OTHER HAND, q_k PROJECTS TO $(s_k(q_i), q_k)$ IN $\pi_A \times \pi_Q$, I.E., $dq_k \rightarrow (ds_k, dq_k)$ IN $d\pi_A \oplus d\pi_Q$. THUS THE BOUNDARY MAP CAN BE WRITTEN AS A 2x2 BLOCK MATRIX $\begin{pmatrix} dr & 0 \\ ds & 1 \end{pmatrix}$,

WHERE $dr: dR \rightarrow d\pi_A$ IS THE BOUNDARY MAP IN $C_T(\tilde{W})$, AND

$ds: d\pi_Q \rightarrow d\pi_A$ IS THE MAP CORRESPONDING TO THE WORDS $s_k(q_i)$.

USING THE ABOVE REWRITING OF THE CO-TANGENT COMPLEX GIVES US

THE FOLLOWING DIAGRAM WHICH WILL GIVE US THE ISOMORPHISM WITH REIDEMEIER

FIGURE 1:

$$\begin{array}{ccccccc}
 0 & \rightarrow & 0 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & d\pi_Q & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & dR \oplus d\pi_Q & \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} & dR & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & d\pi_Q & \xrightarrow{\begin{pmatrix} ds & 1 \end{pmatrix}} & d\pi_A \oplus d\pi_Q & \xrightarrow{\begin{pmatrix} -ds & 1 \end{pmatrix}} & d\pi_A & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & 0 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 0
 \end{array}$$

COLUMN 2 IS THE CO-TANGENT COMPLEX AND COLUMN 3 IS THE COMPLEX $L_p^* \otimes C_T(\tilde{W})$.

D

THAT 'D' COMMUTES IS OBVIOUS EVERYWHERE BUT THE LOWER RIGHT SQUARE.

BUT TO SEE THAT IT COMMUTES TOO, WE MUST SEE JUST THAT

$$\begin{array}{ccc}
 d\pi_Q & \xrightarrow{ds} & d\pi_A \\
 & \searrow & \downarrow \epsilon \\
 & & \Sigma\pi
 \end{array}$$

COMMUTES, I.E. THAT $\begin{array}{ccc} \pi_Q & \xrightarrow{f} & \pi_A \\ & \searrow & \downarrow \epsilon \\ & & \Sigma\pi \end{array}$ COMMUTES.

THE LATTER IS TRUE SINCE E_A IS DEFINED BY $E_A(\alpha) = (\text{IMAGE OF } \alpha \text{ UNDER } \pi) - 1$
 AND LIKEWISE FOR E_ϕ , SO DIAGONAL COMMUTES BY COMMUTATIVITY OF $\pi \circ \phi \rightarrow \pi \circ \alpha$
 $\searrow \downarrow$
 π 51.

THE ROWS OF D ARE ALSO OBVIOUSLY EXACT EXCEPT PERHAPS ROW 3.
 BUT IN THAT ROW, THE EXACTNESS AT THE ENDS IS OBVIOUS, AND
 THE COMPOSITION $(ds, 1) \begin{pmatrix} 1 \\ -ds \end{pmatrix} = ds - ds = 0$, SO WE NEED ONLY
 SHOW THAT $(\alpha, \beta) \in \text{Ker}(ds) \Rightarrow (\alpha, \beta) \in \text{Im}(ds, 1)$. BUT
 WE HAVE $\alpha - ds(\beta) = 0$ I.E. $\alpha = ds(\beta)$, SO $(ds, 1)$ APPLIED TO β
 GIVES $(ds(\beta), \beta) = (\alpha, \beta)$, AS DESIRED.

HENCE THE THREE COLUMNS OF D FORM A SHORT EXACT
 SEQUENCE OF CHAIN COMPLEXES. THE FIRST COLUMN IS OBVIOUSLY
 EXACT, AND THE SECOND IS ALSO EXACT (REMEMBER, IT IS TORSIONED WITH $L^{\pm 1}$),
 BY THE HYPOTHESIS THAT WE ARE AT A TRANSVERSE REPRESENTATION P .
 THUS THE LAST COLUMN, ~~IS ALSO EXACT~~, GIVING THE HOMOLOGY
 OF W WITH $L^{\pm 1}$ COEFFICIENTS, IS ALSO EXACT, AND ALL THE TORSIONS
 ARE DEFINED. THE TORSION t_p OF THE LAST COLUMN IS BY
 DEFINITION THE TORSION OF W WITH $L^{\pm 1}$ COEFFICIENTS, AND AS WE
 SAW IN SECTION X, THE TORSION OF THE MIDDLE COLUMN IS
 $(-1)^{d(g-1)} t_p$. THE TORSION OF COLUMN 1 IS CLEARLY 1.

WE WANT TO APPLY PROP. 21 TO GET THE RELATION
 BETWEEN t_p AND t_p . TO DO THIS WE MUST VERIFY THAT THE
 ROWS ARE "VOLUME-EXACT", I.E., THE VOLUME IN THE CENTER IS
 THE "PRODUCT" OF THOSE ON THE LEFT AND RIGHT. THE FIRST
 AND LAST ROWS ARE OBVIOUS - AS FOR THE SECOND AND THIRD ROWS,
 ONE CAN VERIFY, BY TRACING THROUGH THE DEFINITIONS OF THE
 CORRECT ORIENTATIONS, THAT IF WE ORDER THE P_i 'S AND Q_i 'S SO
 THAT dP_1, \dots, dP_g AND dQ_1, \dots, dQ_g GIVE CORRECTLY PAIRED
 CELLULAR VOLUMES OF dR AND dR_X (AS DEFINED IN VII. F, H)
 THEN ~~THE~~ $da_1, \dots, da_g, dg_1, \dots, dg_g$ GIVES THE CANONICAL VOLUME

ON $dT_A \oplus dT_B$ (AS DEFINED IN X.A), WHERE THE dg_i 's ARE THE DUALS TO THE dP_i 's — I.E. $dP_1, \dots, dP_g, dg_1, \dots, dg_g$ IS \mathbb{S}^2 THE NATURAL VOLUME ON $dT = dR \oplus dT_B$.

IN THIS CASE, IF WE CHOOSE THE VOLUME ON THE dT_B 's IN COLUMN 1 (WHICH IS ~~FREE~~ AT OUR DISPOSAL) TO BE $(-1)^{dg}$ TIMES THE VOLUME GIVEN BY THE ORDERING dg_1, \dots, dg_g , THEN THE VOLUME IN ROWS 2, 3 ARE INDEED "EXACT". THIS IS OBVIOUS IN ROW 2, WHICH ALSO SHOWS WHY THE SIGN WAS CHOSEN THE WAY IT WAS: THE MIDDLE TERM $dR \oplus dT_B$ IS "BACKWARDS" FROM THE WAY WE NORMALLY HAVE FOR THE PRODUCT VOLUME, WHICH HAS dT_B FIRST, dR SECOND. AS FOR ROW 3, LIFT $da_i \in dT_A$ TO $(da_i, 0)$ IN THE DIRECT SUM; THEN A PRODUCT BASIS IS GIVEN BY $(-1)^{dg} \{ \{ (ds_i, dg_i) \}_{i=1}^g, \{ (da_i, 0) \}_{i=1}^g \}$
 $= \{ \{ (da_i, 0) \}, \{ (ds_i, dg_i) \}$, WHICH IS THE SAME VOLUME (GIVING $2 \times \text{dim} - \text{dim}$) AS $\{ \{ (da_i, 0) \}, \{ (0, dg_i) \} \}$ — I.E. THE EXISTING VOLUME ON $dR \oplus dT_B$.

WE CAN NOW APPLY PROPOSITION 21 AND GET

$(-1)^{dg(i-1)} t_p = 1 \cdot T_p \cdot (-1)^{\beta_i \beta_{i+1}}$. TO FIND THE β 'S, WE WRITE DOWN A TABLE OF THE DIMENSIONS IN DIAGRAM D:

DEGREE			
3	0	d	d
2	gd	2gd	gd
1	gd	2gd	gd
0	0	d	d

THEN, USING THE EXACTNESS OF THE COLUMNS AND THE FACT THAT

$\beta'_i = \text{DIM } \mathcal{C}'_i = \text{DIM } \mathcal{C}'_E - \beta'_{i+1}$, THE TABLE OF THE β 'S IS

DEGREE	β'	β	β''
3	0	d	d
2	gd	(g-1)d	(g-1)d
1	0	d	d

HENCE $\sum_i \beta_i \beta_{i+1}'' = \beta_1 \beta_2'' + \beta_2 \beta_3'' = 0 + g d^2 \equiv g d \pmod{2}$

AND WE GET $(-1)^{d(g-1)} t_p = (-1)^{dg} \tau_p$. So FINALLY; 53.

AFTER ALL THOSE PROPOSITIONS, WE FINALLY HAVE A

THEOREM 1: THE TANGENTIAL "TORION" t_p AT A TRANSVERSE REPRESENTATION ρ OF $\pi_1(W^3)$ IS $(-1)^{dg} \tau_p$, WHERE τ_p IS THE REIDEMEISTER TORSION OF W WITH L_p COEFFICIENTS AND d IS THE DIMENSION OF THE LIE ALGEBRA L .

XII. $\Omega(2)$ TORSION OF KNOTS AND SURGERY MANIFOLDS

A. HAVING PROVED THAT t_p IS JUST THE REIDEMEISTER TORSION OF W USING L_p COEFFICIENTS, WE NOW TAKE A NEW TACK. OUR MAIN OBJECT IS TO PRODUCE INVARIANTS OF W BY FORMING THE POLYNOMIAL

$T(t) = \prod (t - t_p)$, THE PRODUCT BEING TAKEN OVER ALL TRANSVERSE p , I.E. OVER ALL ISOTROPIC p WITH $t_p \neq 0$. FORMULATED THIS WAY, IT IS NO LONGER SEEM PARTICULARLY PRESSING OR NATURAL TO USE L_p COEFFICIENTS. IN FACT, IF WE ARE GIVEN ANY REPRESENTATION OF THE

LIE GROUP G ON A (COMPLEX) VECTOR SPACE V , WE GET A CO-MODULE V_p FOR ANY $\rho: \pi \rightarrow G$, AND WE CAN REASONABLY DEFINE THE TORSION τ_p AND THE POLYNOMIAL $\prod (t - t_p)$. SEVERAL PROBLEMS ARISE. FIRST, τ_p MAY NOT BE DEFINED, I.E., WE MAY HAVE $H_*(W, V_p) \neq 0$.

BUT BY ANALOGY WITH L_p COEFFICIENTS, WHOSE $t_p = 0$ IFF $H_* \neq 0$, WE DEFINE $\tau_p = 0$ WHEN $H_*(W, V_p) \neq 0$, AND TAKE THE PRODUCT OVER JUST THE REPS p WHICH ARE ACYCLIC, I.E. $H_*(W, V_p) = 0$.

SECOND, IT CAN HAPPEN THAT τ_p IS DEFINED AND NON ZERO FOR AN INFINITE SET OF REPRESENTATIONS — IN FACT, A WHOLE SUBALGEBRA OF THE REPRESENTATION SPACE OF $\pi_2(W)$, OF POSITIVE DIMENSION. TO DEAL WITH THIS WE RESTRICT OUR ATTENTION TO THE 0-DIMENSIONAL SUBSET $R_0(W)$ OF THE REPRESENTATION SPACE; IT IS AN ALGEBRAIC SET (OVER \mathbb{C}) AND HENCE FINITE. THUS $\prod_{(p \in R_0(W))} (t - \tau_p)$ IS A WELL DEFINED

POLYNOMIAL. THIS "EXCLUSIVE" DEFINITION OF THE PERMISSIBLE p 'S IS SOMEWHAT UNNATURAL LOOKING — BUT IT DOES PRODUCE SOME INTERESTING POLYNOMIAL INVARIANTS OF 3-MANIFOLDS.

WE NOW AND HOWEVER RESTRICT OUR ATTENTION TO $G = \Omega(2, \mathbb{C})$ AND $V = \mathbb{C}^2$ THE STANDARD REP OF $\Omega(2)$. IT TURNS OUT THAT THE COMPUTATION OF THE TORSION IS IN MOST CASES MUCH EASIER USING V_p COEFFICIENTS THAN WITH L_p COEFFICIENTS, AND THE RESULTING

POLYNOMIAL SIMPLER THAN THE CORRESPONDING POLYNOMIAL FOR L_p .

Also, since $\dim V_p$ is even, we don't have to worry about the sign determination problem.

55

We mentioned that the space $\hat{R}_0(W)$ is an algebraic set defined over \mathbb{Q} ; also, the torsion is an algebraic function on $\hat{R}_0(W)$, and defined over \mathbb{Q} . Thus the torsion consists of algebraic numbers, and the polynomial $\prod (t - \tau_p)$ has rational coefficients. It was found empirically in the cases quoted that the τ_p 's were algebraic integers and in fact were divisible by 2, both with L_p and V_p coefficients. Division by 2 simplifies the polynomial appreciably, so we will work with the "half torsion" $s_p = \frac{1}{2}\tau_p$, and denote the V_p torsion polynomial by $\sum W(s) = \prod (s - s_p)$. This division by 2 is very mysterious at the moment. It parallels Casson's division by 2, but I emphasize that there is no reason for it a priori; it just simplifies the known examples, and it is not clear that all V_p -torsion is an algebraic integer times 2.

The primary reason that s_p is easier to compute than τ_p (i.e. L_p coefficients) is that for many knot components the torsion is defined on the whole space \hat{R} of rep classes of the knot group, and is in fact an algebraic function τ_p on \hat{R} . In this case, when W is a surgery manifold of the knot and $p \in \hat{R}$ is a suitable representation ~~which~~ which extends to W , then there is a simple formula connecting s_p and τ_p . The goal of this chapter is to establish this formula and write some of the polynomials \int .

B. A PARABOLIC ELEMENT $\neq I(2)$ IS A NONTRIVIAL ELEMENT WHICH FIXES SOME (NON-ZERO) VECTOR, OR EQUIVALENTLY, AN ELEMENT CONJUGATE TO $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ ($t \neq 0$). NOTE THAT: 56:

a) THE SET OF FIXED VECTORS FORM A LINE ($\neq \emptyset$).

b) IF THE ELEMENT IS g , THEN $g w - w = (g-1)w = u + w(v)$ FOR ALL $w \in V$.

WE NOW EXAMINE THE REPS OF THE 2-DIMS INTO $SL(2)$.

DEF. A REP ρ OF $\pi_1(T^2)$ IS PARABOLIC IF $\rho(x)$ IS EITHER TRIVIAL OR PARABOLIC FOR ALL $x \in \pi_1(T^2)$.

PROP. 23 THE FOLLOWING ARE EQUIVALENT:

a) ρ IS PARABOLIC.

b) THE MATRICES $\rho(x)$ ALL HAVE A COMMON FIXED (NON-ZERO) VECTOR.

c) $\det(\rho(x) - 1) = 0$ FOR ALL $x \in \pi_1(T^2)$.

PROOF: IF ρ IS TRIVIAL, SO IS THE PLAN, SO ASSUME ρ IS A NONTRIVIAL REP.

a) \Rightarrow b): LET $\rho(x) \neq 1$ BE PARABOLIC, FIXING $v \neq 0$. LET $y \in \pi_1(T^2)$ BE ANY OTHER ELEMENT SUCH THAT $\rho(y) \neq 1$. SINCE $\rho(y)$ COMMUTES WITH $\rho(x)$, IT HAS A COMMON EIGENVECTOR, WHICH CAN ONLY BE v AND ITS MULTIPLES. BUT $\rho(y)$ IS PARABOLIC WRT TO ITS EIGENVALUES! $\neq 1$, I.E. y DOES NOT FIX v .

b) \Rightarrow c): ALL THE MATRICES HAVE EIGENVALUE 1 , SO $\det(\rho(x) - 1) = 0$, $\forall x$.

c) \Rightarrow a): ALL $\rho(x)$ HAVE EIGENVALUE 1 , SO FIX A VECTOR - I.E., THEY ARE EITHER TRIVIAL OR PARABOLIC.

IN THE FOLLOWING PROPOSITIONS WE WILL FREQUENTLY WRITE x FOR BOTH AN ELEMENT OF π_1 AND FOR ITS CORRESPONDING MATRIX $\rho(x)$, I.E. WE ABBREVIATE $\rho(x)$ TO x .

PROP. 24 LET ρ BE A REP $\cong \pi_1(T^2) \rightarrow SL(2)$. THEN $H_x \left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \frac{1}{\rho} \right) = 0$ IFF ρ IS NONPARABOLIC.

PROOF: a) LET ρ BE NONPARABOLIC. THEN THERE IS A PRIMITIVE ELEMENT $x \in \pi_1(T^2)$ SUCH THAT $\det(x-1) \neq 0$, I.E. $x-1$ IS INVERTIBLE. LET $y \in \pi_1(T^2)$ BE DUAL TO x , I.E. $x \cdot y = 1$. WE COMPUTE H_x USING

THE COMPLEX OF III, B:

$$0 \rightarrow V \xrightarrow{\alpha \pm \beta} V^2 \xrightarrow{\epsilon} V \rightarrow 0 \quad \text{WHERE } Z = [x, y]$$

$$dZ = (x-1)dy - (y-1)dx$$

57.

i.e.

$$0 \rightarrow V \xrightarrow{\partial_2} V^2 \xrightarrow{\partial_1} V \rightarrow 0$$

SINCE $\alpha-1$ IS INVERTIBLE, ∂_1 IS ONTO. AND $\text{Ker } \partial_1$ IS 2-DIMENSIONAL.
 LIKEWISE, ∂_2 IS 1-1 AND HAS 2-DIMENSIONAL IMAGE. THIS PROVES
 THAT THE COMPLEX IS ACYCLIC.

b) LET ρ BE PARABOLIC. THEN α, β HAVE A COMMON FIXED VECTOR $\tau \neq 0$.
 NOW $(\alpha-1)\tau$ IS IN THE LINE (τ) FOR ALL $\tau \in V$, AND LIKEWISE FOR
 $(\beta-1)\tau$. HENCE $\text{Im} \begin{pmatrix} \alpha-1 \\ \beta-1 \end{pmatrix}$ IS ALSO CONTAINED IN THIS LINE, AND IN
 PARTICULAR ∂_1 IS NOT ONTO, I.E. $H_0 \neq 0$.

THE ABOVE SUGGESTS THAT WE COMPUTE THE TORSION $\tau(T^2, V_\rho)$ FOR
 NON-PARABOLIC ρ .

PRIP. 25 FOR NON-PARABOLIC ρ , $\tau(T^2, V_\rho) = 1$

PROOF: LET e_1, e_2 BE A BASIS FOR V AND α, β A GEOMETRIC BASIS
 FOR $\pi_1(T^2)$ SUCH THAT $\alpha-1$ IS AN INVERTIBLE MATRIX, AS IN THE PREVIOUS
 PROPOSITION. TO SEE THAT $\tau = 1$ IS THE SAME AS SHOWING THAT THE
 "CELLULAR" VOLUME ON U^2 GIVEN BY THE CELLULAR BASIS

$$(e_1, 0), (e_2, 0); (0, e_1), (0, e_2)$$

IS THE SAME AS THE PRODUCT OF THE VOLUME ON THE TWO ENDS V, V .
 BUT THE PRODUCT VOLUME HAS BASIS

$$\underbrace{((-y-1)e_1, (x-1)e_1), (-y-1)e_2, (x-1)e_2)}_{\text{IMAGE OF } C_2 = V} ; \underbrace{((x-1)e_1, 0), ((x-1)e_2, 0)}_{\text{LIFT OF } C_1 = V}$$

A MULTIPLICATION OF THE LAST TWO VECTORS BY $(\alpha-1)$ AND THE FIRST TWO BY $(\beta-1)$,
 WHICH IS VOLUME PRESERVING, GIVES BASIS $(\alpha_1, \alpha_2), (\beta_1, \beta_2); (e_1, 0), (e_2, 0)$.

AND NOW IT IS CLEAR THAT ADD OPERATIONS CONVERT THIS INTO

$$(e_1, 0), (e_2, 0), (0, e_1), (0, e_2), \dots, \emptyset \in D.$$

58.

WE HAVE A SIMILAR SITUATION FOR A 1-D TORUS $S = S^1 \times D^2$ WITH

$\pi_1 = \mathbb{Z}$ GENERATED BY χ : $H_1 = 0$ IFF $\rho(\chi)$ IS NOT PARALLEL (OR TRIVIAL)

PROP 26 IF ρ IS A NONPARALLEL REP $\neq \pi_1(S)$, I.E. $\rho(\chi) \neq \pm 2\pi$ PARALLEL,
 THE TORSION $\tau(S, V_\rho) = \det^{-1}(\rho - 1)$

PROOF: THE CHAIN COMPLEX IS

$$0 \rightarrow V \xrightarrow{\chi - 1} V \rightarrow 0 \quad \text{AND } \chi - 1 \text{ MULTIPLIES VOLUME BY } \frac{1}{2\pi}(\chi - 1).$$

THE TORSION IS $\text{VOL}_0 / \text{VOL}(\text{Im}(\chi - 1)) = \det^{-1}(\chi - 1).$

THE FOLLOWING PROPOSITION GIVES US THE CONNECTION BETWEEN THE TORSION OF SUB-COMPLEMENTS AND 3-MANIFOLDS.

PROP 27 LET W BE GIVEN BY A TORUS DECOMPOSITION $A \cup_{T^2} B$, AND LET ρ BE A REPRESENTATION OF $\pi_1(W)$ WHICH RESTRICTED TO T^2 IS NONPARALLEL. THEN $H_1(W) = 0$ IFF $H_1(A) = H_1(B) = 0$, AND IN THIS CASE

$$\tau(W, V_\rho) = \tau(A, V_\rho) \tau(B, V_\rho).$$

PROOF: BY THE MIXED-VIETORIS FORMULA, $H_1(W) = 0$ IFF $H_1(A) \oplus H_1(B) = 0$. IN THIS CASE WE USE THE EXACT SEQUENCE OF COIN COMPLEXES.

$$0 \rightarrow C_\rho(T^2, V_\rho) \xrightarrow{(i_A - i_B)} C_\rho(A, V_\rho) \oplus C_\rho(B, V_\rho) \xrightarrow{\text{JUN}} C_\rho(W, V_\rho) \rightarrow 0$$

AND PROP. 21 TO SHOW THAT $\tau(A) \tau(B) = \tau(T^2) \tau(W) = \tau(W)$ (SINCE $\tau(T^2) = 1$)

WE MUST ONLY VERIFY THAT THE LHS ARE "VOLUME FACTS". BUT THE VOLUMES AND MASS ARE RELATIVE, SO THIS IS OKAY, EXCEPT PERHAPS THE SIGN, AND THE SIGN IS NO PROBLEM SINCE ALL THE DIMENSIONS (BOUNDARIES INCLUDED) ARE EVEN.

WE HAVE THE FOLLOWING EASY COROLLARY:

PROP. 28 Let: K be a knot component and W the 3-manifold
 got by $\frac{1}{n}$ surgery on K , i.e. we kill the curve $m \ell^n$ on $T^2 = \partial K$
 where m, ℓ are the knot meridian and longitude. Let ρ be
 an acyclic rep. of $\pi_1(W)$, non-parabolic on ∂K . If τ_ρ is the torsion
 of K at ρ (defined by the previous proposition), then

59.

$$\tau(W, V_\rho) = \tau_\rho \cdot \det(\ell-1) = \tau_\rho / \text{Tr}(\ell-1) = \tau_\rho / (2 - \text{Tr} \ell)$$

PROOF: - we use the previous proposition with $A = K, B = \text{solid torus}$.
 B has a link $m \ell^n$ which kills $m \ell^n$ on $\partial B = \partial K$, and the
 curve ℓ intersects $m \ell^n$ once, so ℓ separates $\pi_1(B)$.

- By proposition 26, $\tau(B, V_\rho) = \det^{-1}(\ell-1)$ which shows that

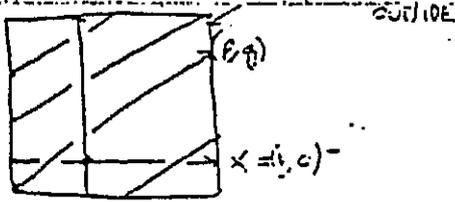
$$\tau(W, V_\rho) = \tau_\rho / \det(\ell-1) \quad \text{THE OTHER EQUALITIES FOLLOW FROM}$$

$$\det(\ell-1) = (\lambda-1)(\lambda^{-1}-1) = 2 - \text{Tr} \ell, \text{ where } \lambda \text{ is an eigenvalue of } \ell$$

THE ABOVE REDUCES THE DETERMINATION OF THE TORSION OF $\frac{1}{n}$ SURGERY MANIFOLDS
 TO THE DETERMINATION OF THE KNOT TORSION AND THE LONGITUDINAL TRACE.

- WE SHOULD MENTION THAT THE REQUIREMENT $\rho|_{T^2}$ IS NON-PARABOLIC ALMOST
 ALWAYS HELD — IN FACT, GIVEN THAT WE ARE ONLY LOOKING AT THE
 0-DIMENSIONAL SET OF $\hat{R}(W)$, IT MAY ALWAYS HOLD IN ANY CASE,
 IF THERE WERE A PARABOLIC ρ IN $\hat{R}(W)$, AND $H_2(W, V_\rho) \text{ WERE } 0$,
 IT IS STILL POSSIBLE TO COMPUTE ITS TORSION — WE JUST CAN'T USE PROP. 28.

C. NEXT, WE CONSTRUCT THE SPACE \hat{R} OF REPS OF THE (β, β) TORUS KNOT.
 LET $T^2 \subset S^3$ HAVE LONGITUDE $(1, 0)$ AND MERIDIAN $(0, 1)$ SUCH THAT
 $X = (1, 0)^-$ GENERATES π_1 OF THE "INSIDE", WHERE $-$ MEANS PUSH THE CURVE
 INSIDE, AND $Y = (0, 1)^+$ GENERATES THE "OUTSIDE". WE DEPICT
 THIS IN RELATION TO THE TORUS AND THE KNOT (β, β) LIKE THIS:



- NOTE THAT X IS HOMOTOPIC TO $(1, b)^+$ FOR ANY b AND $Y \sim (a, 1)^+$ FOR ANY a .
 IN PARTICULAR WE HAVE $X^p = (p, 0)^- = (p, q)^- = (p, q)^+ = (0, q)^+ = Y^q$,
 SINCE WE CAN MOVE $(p, q)^-$ FROM INSIDE, PAST THE KNOT TO THE OUTSIDE $(p, q)^+$
 [IT IS EASY TO SEE THAT $\int^Z (p, q)$ ORIENTATION RETRACTS ONTO THE
 UNION OF THE ONE-HANDLES X, Y AND THE 2-CELL ATTACHED BY $X^p Y^{-q}$,
 SO THAT THE KNOT GROUP HAS PRESENTATION $\langle X, Y \mid X^p = Y^q \rangle$.

ALSO $Z = X^p = Y^q$ IS PERIPHERAL: WE CAN SLIDE $X^p = (p, q)^-$ UP
 TO A PARALLEL COPY OF THE KNOT. NOTE THAT Z COMMUTES WITH
 X AND Y AND SO IS CENTRAL. LET NOW $r, s \in \mathbb{Z}$ BE CHOSEN SUCH
 THAT $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = 1$: THEN THE CURVE (r, s) ON THE TORUS INTERSECTS

(p, q) ONCE. THE CURVE $(r, s)^- = (r, 0)^- = X^r$ LOOKS LIKE THIS:



THE CURVE $(r, s)^+ = (0, s)^+ = Y^s$ LOOKS THE SAME, BUT PASSES OVER THE KNOT.

HENCE $m = X^{-r} Y^s$ IS A MERIDIAN OF THE KNOT.



NOTE THAT THE INTERSECTION $m \cdot Z = \pm 1$ SINCE Z
 GOES ONCE AROUND THE PERIPHERAL TORUS, THE "LONG" WAY.

NOW IN $H_1 = \pi_1$ OF THE KNOT COMPLEMENT WE CAN WRITE

$$m^p = X^{-pr} Y^{ps} = Y^{-qr} Y^{ps} = Y^{-1} = Y$$

AND HENCE $m^p \delta = Y^{-1} = Z^{-1}$, i.e. Z IS $p\delta$ IN H_1 .

THUS $\ell = Z^{-1} m^p \delta$ IS NULL HOMOLOGOUS AND PERIPHERAL. ALSO
 $\ell \cdot m = Z^{-1} \cdot m = \pm 1$. THEREFORE ℓ IS THE KNOT LONGITUDE.

WE ASSEMBLE THESE FACTS IN

PROP. 29

LET $\xi, \delta \in \mathbb{Z}$ SATISFY $\left| \begin{pmatrix} \xi & 1 \\ 1 & \delta \end{pmatrix} \right| = 1$; THEN

- a) $Z = X^\xi = Y^\delta$ IS CENTRAL AND PERIPHERAL
 b) $m = X^{-\xi} Y^\delta$ IS A KNOT MERIDIAN
 c) $l = Z^{-1} m^{\xi\delta}$ IS A CORRESPONDING KNOT LONGITUDE.

61.

IT WILL BE USEFUL TO INTRODUCE THE ELEMENTS $U = X^\xi, V = Y^\delta$, SO THAT $m = UV$.

THEN

PROP. 30

a) $X = Z^{\xi} U^{\delta}$

b) $Y = Z^{-\xi} V^{\delta}$

PROOF: $Z^{\xi} U^{\delta} = Z^{\xi} X^{-\xi\delta} = Z^{\xi} X^{-\xi\delta} = Z^{\xi} X Z^{-\xi} = X$
 & SIMILARLY FOR b).

PROP 31

LET $\pi = \pi_1(S^3 - p, \delta)$ AND LET ρ BE AN IRREDUCIBLE REP OF π INTO $GL(2, \mathbb{C})$; THEN (IDENTIFYING AGAIN $\rho(X)$ WITH X , ETC.):

a) $Z = \pm 1, X^{2\xi} = Y^{2\delta} = 1$

b) $X^{\xi} \neq \pm 1, Y^{\delta} \neq \pm 1$

PROOF: a): SUPPOSE $Z \neq \pm 1$ AND LET ν BE AN EIGENVECTOR OF Z ; SINCE Z COMMUTES WITH X & Y , X & Y MUST HAVE ν AS AN EIGENVECTOR, CONTRADICTION THE IRREDUCIBILITY OF ρ . THUS $Z = \pm 1$. THEN $X^{2\xi} = Y^{2\delta} = 1$ FOLLOWS IMMEDIATELY.

b) SUPPOSE $X^{\xi} = \pm 1$; THEN $X^{2\xi} = 1$ AND $X^{2\xi} = 1$. SINCE $(\xi, \delta) = 1$, THIS IMPLIES $X^2 = 1, X = \pm 1$, AGAIN CONTRADICTION THE IRREDUCIBILITY. $Y^{\delta} \neq \pm 1$ IS PROVED SIMILARLY.

THUS AN IRREDUCIBLE REP OF π GIVES X, Y EIGENVALUES WHICH ARE POWERS OF UNITY, NOT ± 1 , SAY $\alpha = e^{\frac{2\pi i a}{p}}, \alpha^{-1}$ FOR X AND $\beta = e^{\frac{2\pi i b}{\delta}}, \beta^{-1}$ FOR Y , AND WE CAN ASSUME THAT

$0 < a < p, 0 < b < q$. Note also that $X^p = (-1)^a = Z = Y^q = (-1)^b$
 so we must have $a \equiv b \pmod{2}$. Conversely, given any such a, b
 and any X, Y with these eigenvalues we get a rep of π . 62.

Prop 32 Let $\text{Tr } X, Y$ be fixed \mathbb{C} values corresponding to a choice
 of α, β as described above, and let $t \in \mathbb{C}$. Then there

is a representation ρ such that $\text{Tr } \rho = t$. If
 $t \neq \alpha^{-r} \beta^s + \alpha^r \beta^{-s}$ & $\alpha^r \beta^s + \alpha^{-r} \beta^{-s}$, i.e. ~~neither~~ $\neq 2 \cos \pi \left(\frac{ar}{p} \pm \frac{bs}{q} \right)$,
 then ρ is irreducible.

Proof: Let $X = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ and $Y_0 = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$: then we can choose
 Y to be any conjugate of Y_0 . Now $X^{-r} = \begin{pmatrix} \alpha^{-r} & 0 \\ 0 & \alpha^r \end{pmatrix}$ and
 Y^s is any conjugate of $\begin{pmatrix} \beta^s & 0 \\ 0 & \beta^{-s} \end{pmatrix}$. By choice of $\beta, \beta^s \neq \pm 1$, and hence
 Y^s can be any matrix in $SL(2)$ of the form $\begin{pmatrix} \beta^s + u & u \\ u & \beta^s - u \end{pmatrix}$
 (it has the right trace) for any $u \in \mathbb{C}$.

Then ~~we have~~ $\rho = X^{-r} Y^s$ has trace $\alpha^{-r} \beta^s + \alpha^r \beta^{-s} + u(\alpha^{-r} - \alpha^r)$
 by choice, $\alpha^{-r} - \alpha^r \neq 0$, so we can choose u to get $\text{Tr } \rho = t$.

To see irreducibility, let the representation be reducible. Conjugate
 so that the common eigenspace is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$; then $X = \begin{pmatrix} \alpha^r & A \\ 0 & \alpha^{-r} \end{pmatrix}$ and
 $Y = \begin{pmatrix} \beta^{-s} & B \\ 0 & \beta^s \end{pmatrix}$ where the \pm signs in X are independent of those in Y .

Computing the trace of $X^{-r} Y^s$ gives the two possible values
 described in the ~~previous~~ proposition.

Prop. 33 Two irreducible reps of π with the same values of
 $\text{Tr } X, \text{Tr } Y$ and $\text{Tr } \rho$ are conjugate.

Proof: $\text{Tr } X$ determines the trace of its powers, e.g. $\text{Tr } Z$ and $\text{Tr } U$,
 and since $Z = \pm 1$, determines Z . Likewise $\text{Tr } Y$ determines
 $\text{Tr } V$. Hence the given data determines $Z, \text{Tr } U, \text{Tr } V, \text{Tr } \rho = \text{Tr } UV$
 The ~~two~~ matrices U, V are also irreducible (they are at unit wick 2 in $\{X, Y\}$).

MISSING ONLY THE CASE $\zeta = \pm 1$, SO THE DATA DETERMINES U, V UP TO MUTUAL CONJUGATION. UNLESS THE FORMULA OF PROP 30 SHOWS THAT X, Y ARE DETERMINED UP TO MUTUAL CONJUGATION, QED.

WE SUMMARIZE THE PRECEDING TWO PROPS BY

63.

Prop 34 THE COMPONENTS OF IRREDUCIBLE REPRESENTATIONS OF $\pi = \pi_1(S^3 - (p, q))$ ARE GIVEN BY $\hat{R}_{(a,b)}$ WHERE:

- a) $0 < (a, b) < (\bar{p}, \bar{q})$ AND $a \equiv b \pmod{2}$
- b) $\zeta = (-1)^a$
- c) $\hat{R}_{(a,b)}$ IS ALL EIGENVALUES WITH $\text{Tr } X = 2 \cos \frac{\pi a}{p}$, $\text{Tr } Y = 2 \cos \frac{\pi b}{q}$
- d) $\text{Tr } M$ TAKES ALL VALUES ON $\hat{R}_{(a,b)}$ AND IS ± 1 ON IRREDUCIBLE
- e) THE REP. IS IRREDUCIBLE UNLESS $\text{Tr } M = 2 \cos \frac{\pi a}{p} = \pm \frac{2b}{q}$

NEEDN'T

D. NEXT WE FIND THE REPRESENTATIONS OF THE $\frac{1}{N}$ SURFACE MANIFOLD W_N OF THE (p, q) TORUS KNOT. RECALL THAT (PROP 28) $\lambda = \zeta^{-1} \mu^q$, WHERE λ IS THE LENGTH OF THE KNOT, $\pi_1(W_N)$ IS GIVEN BY ADDING THE RELATION $\mu^N = 1$, WHICH, AS FAR AS $\Omega(2)$ RELATIONS GOES, GIVES US $\mu^{p(N+1)} \zeta^{-N} = 1$. I.E. $\mu^{p(N+1)} = \zeta^N$.

CONVERSELY IF μ SATISFIES THE LATTER THEN IT'S EASY TO SEE $\mu^N = 1$ IN $\Omega(2)$.

PROF: $N = |pq(N+1)|$, THEN μ HAS EIGENVALUES

$$\mu = e^{\frac{2\pi i M}{N}}; \mu^{-1}, \text{ WHERE } 0 < M < N \text{ AND } M \equiv Nq \pmod{2}.$$

THE LATTER IS BECAUSE $\mu^N = (-1)^M = \zeta^N = (-1)^{Nq}$ I.E. $M \equiv Nq \pmod{2}$

(ALSO, M CANNOT BE 0 OR N , SINCE THIS $\Rightarrow \mu = \pm 1$, CHIRALITY NOT IRREDUCIBLE)

HENCE:

CLASS OF

Prop 35 THE DISTINCT IRREDUCIBLE REPRESENTATIONS OF THE SURFACE

MANIFOLD W_N ARE GIVEN BY (a, b, M) , WHERE

- a) $0 < (a, b) < (\bar{p}, \bar{q})$ AND $a \equiv b \pmod{2}$
- b) $0 < M < N = |pq(N+1)|$ AND $M \equiv Nq \pmod{2}$
- c) $\text{Tr } X = 2 \cos \frac{\pi a}{p}$, $\text{Tr } Y = 2 \cos \frac{\pi b}{q}$, $\text{Tr } M = 2 \cos \frac{\pi M}{N}$

(245)

PROOF: THE REPRESENTATION Λ EXISTS AND IS UNIQUE BY PROPS 32, 33 AND ONLY ARE DISTINCT SINCE ENERGY ARE 10. WE NEED ONLY SHOW THEY ARE IRREDUCIBLE. BUT $\frac{1}{f} \pm \frac{1}{g}$ IS NEVER CONGRUENT TO $\frac{1}{N}$ MOD 2, SINCE N IS RELATIVELY PRIME TO f, g . THUS $\text{Tr } m \neq 2 \text{Tr } \left(\frac{1}{f} \pm \frac{1}{g} \right)$ AND IRREDUCIBILITY FOLLOWS FROM PROP 32. 64.

E. NEXT WE COMPUTE H_2 OF THE BOLS THAT WITH V_p COEFFICIENTS AND THE TORSION τ_p WHEN $H_X = 0$. WE USE THE \mathbb{R} -COMPLEX

$$0 \rightarrow \overline{\mathbb{R}} \xrightarrow{d(x^p, y^q)} \overline{\mathbb{R}} \xrightarrow{\text{FREE}} \mathbb{Z} \rightarrow 0 \quad \text{WHERE } \overline{\mathbb{R}} = \text{EVALUATED AT } x, y$$

REMAINING WITH V , AND USING $d(x^p, y^q) = \sum X dx - \sum Y dy$ WHERE $\sum X = 1 + X + \dots + X^{p-1}$, $\sum Y = 1 + Y + \dots + Y^{q-1}$, WE GET

$$0 \rightarrow V \xrightarrow{\begin{pmatrix} \sum X & -\sum Y \end{pmatrix}} V \xrightarrow{\begin{pmatrix} X-1 \\ Y-1 \end{pmatrix}} V \rightarrow 0$$

FOR OUR IRREDUCIBLE REPS, X, Y ARE OF FINITE ORDER AND $X-1, Y-1$ ARE INVERTIBLE. NOW $\sum X$ SATISFIES $(X-1)\sum X = X^p - 1 = Z - 1$, & LIKEWISE $(Y-1)\sum Y = Z - 1$

PROP 36 $H_2(S^3 - (p, q), V_p) = 0$ IFF $\rho(Z) = -1$

PROOF: IF $Z = 1$ THEN $(X-1)\sum X = 0$ SO $\sum X = 0$ & LIKEWISE $\sum Y = 0$.

THEN THE BOUNDARY MAP ∂_2 IS ZERO AND $H_2 = V$.

IF $Z = -1$ THEN $(X-1)\sum X = Z - 1 = -2$, SO $\sum X$ IS INVERTIBLE. HENCE ∂_2 IS 1-1 AND HAS 2-DIMENSIONAL IMAGE. BUT ∂_1 IS SURJECTIVE ($X-1$ IS INVERTIBLE) AND HENCE HAS 2-DIMENSIONAL KERNEL; SO THE COMPLEX IS ACYCLIC.

WE NEED, THEN, COMPUTE THE TORSION ^{ONLY} FOR REPS WITH $Z = -1$, IS $\text{RA } 3; 6$ OK.

PROOF 5.1 ON THE COMPONENT $\tau(a, b)$ THE TORSION HAS THE CONSTANT VALUE

65.

$$\tau(a, b) = \frac{1}{(1 - \cos \frac{\pi a}{p})(1 - \cos \frac{\pi b}{q})}$$

PROOF: FROM THE ABOVE CHAIN COMPLEX, UPI WITH e_1, e_2 OF THE SECOND V TO $(\alpha_1, (Y-1)^{-1}e_1), (\alpha_2, (Y-1)^{-1}e_2)$. ALSO, WITH e_1, e_2 OF THE FIRST V INTO $(\Sigma X(e_1), \alpha_1), (\Sigma X(e_2), \alpha_2)$. THEN τ IS THE VOLUME OF THESE FOUR ELEMENTS, WHICH IS JUST $\det(\Sigma X) \det^{-1}(Y-1)$. BUT WE SAW THAT $\Sigma X = -2(X-1)^{-1}$, SO $\det(\Sigma X) = 4 \det(X-1)^{-1}$.

HENCE $\tau = \frac{4}{\det(X-1) \det(Y-1)} = \frac{4}{(2 - \text{Tr } X)(2 - \text{Tr } Y)} = \frac{4}{(2 - 2\cos \frac{\pi a}{p})(2 - 2\cos \frac{\pi b}{q})}$ QED.

THEOREM 2 THE TORSION OF THE SURFACE REPRESENTATION $\rho(a, b, M)$ ON W_M , WHERE $0 < (a, b, M) < (2\pi, N = |G_M|)$, $a, b \in \mathbb{R}$ AND $M \in \pi \mathbb{Z}$, IS GIVEN BY

$$\tau(a, b, M) = \frac{1}{(1 - \cos \frac{\pi a}{p})(1 - \cos \frac{\pi b}{q})} \cdot \frac{1}{2(1 + \cos \frac{\pi M p q}{N})}$$

PROOF: BY PRO 2F, τ IS GIVEN BY THE FIRST BOUND (I.E. THE KNOT TORSION).

THUS $\det^{-1}(L-1) = \frac{1}{2 - \text{Tr } L}$. BUT $L = Z^{-1} M P Z$ AND $Z = -1$,

SO $L = -M P$. THE EIGENVALUES OF M ARE $e^{\pm \frac{2\pi i k M}{N}}$, SO THE EIGENVALUES OF $-M P$ ARE $-e^{\pm \frac{2\pi i k M p q}{N}}$ AND HENCE $\text{Tr } L = -2 \cos \frac{\pi M p q}{N}$, QED.

WE DO ONE EXAMPLE BY HAND, THE $\sqrt{-1}$ SURFACE ON THE TREFOIL KNOT $(p, q) = (3, 2)$. THE MANIFOLD W_1 IS THE PRINCIPAL SPACE

THERE IS ONLY ONE SUBGROUP: $(a, b) < (3, 2) \Rightarrow a = b = 1$. THE KNOT TORSION ON \hat{R}_3 IS THEN $\frac{1}{(1 - \cos \frac{\pi}{3})(1 - \cos \frac{\pi}{2})} = \frac{1}{(1 - \frac{1}{2})(1 - 0)} = 2$

AS FOR M , WE HAVE $N = |G_M| = 5$ SO M , WHICH MUST BE 001 ($\equiv \pi \cdot 1$),

REPRESENTATION OF $1 \leq 3$. THE LONGITUDE TERM'S DENOMINATOR \neq THE ω

$$2 \left(\text{HOM} \frac{M_{18}}{N} \right) = \left\{ \begin{array}{l} 2 \left(\text{HOM} \frac{6\pi}{5} \right) \\ 2 \left(\text{HOM} \frac{18\pi}{5} \right) \end{array} \right\} = \left\{ \begin{array}{l} 2 \left(1 - \cos \frac{6\pi}{5} \right) \\ 2 \left(1 - \cos \frac{18\pi}{5} \right) \end{array} \right\} = \left\{ \begin{array}{l} \omega^{-2} \\ \omega^2 \end{array} \right\} \quad 66.$$

WHERE ω IS THE "GOLDEN MEAN", $\omega^2 = \frac{3+\sqrt{5}}{2}$ $\omega^{-2} = \frac{3-\sqrt{5}}{2}$

HENCE THE "SEMI-TORSION" $\int_{\rho} = \frac{1}{2} T$ IS ω^2 AND ω^{-2} . AT THE TWO REPRESENTATIONS. THE POLYOM POLYNOMIAL IS THEN $\int(s) = s^2 - 3s + 1$.

THIS IS THE POLYNOMIAL INVARIANT FOR THE POINCARÉ SPHERE.

HERE IS A LIST OF SOME OF THE \int -POLYS FOR OTHER SURFACES ON THE TREFOIL KNOT:

n	COEFFICIENTS, STARTING WITH TOP DEGREE									
-3	1	-36	210	-402	415	-220	91	-15	1	
-2	1	-15	35	-27	9	-1				
-1	1	-3	1							
0	1									
1	1	-5	5	-1						
2	1	-21	70	-124	45	-11	1			
3	1	-45	330	-1224	1287	-1001	455	-120	17	-1

WE NOTE THAT THE POLYNOMIALS ARE MONIC AND HAVE INTEGRAL COEFFICIENTS, AND THE FINAL COEFFICIENTS ARE ± 1 . THUS, THE SEMI-TORSIONS ARE ALGEBRAIC INTEGERS, AND ARE UNITS IN THE RING OF ALG. INTEGERS. WE CAN ASK IF THIS IS ALWAYS TRUE FOR \int_{ρ} COEFFICIENTS (NOTE: IT FAILS FOR L_{ρ} COEFFICIENTS!).

THE FOLLOWING RECURSION FORMULA WAS FIRST NOTED EMPIRICALLY:
 LET $\overline{\int}_n$ BE THE POLY \int_n WITH REVERSED COEFFICIENTS, I.E. ITS ROOTS ARE $\frac{1}{s_0}$.
 NORMALIZE $\overline{\int}_n$ SO THAT THE CONSTANT TERM IS $(-1)^n$. THEN THE FOLLOWING RECURSION FORMULA HELDS.

$$\overline{\int}_{n+1} = D(s) \overline{\int}_n - \overline{\int}_{n-1}, \quad \text{WHERE } D(s) = s^3 - 6s^2 + 9s - 2$$

WE SHOW ABOVE THIS FORMULA.

F. WE FIRST DERIVE AN EXPRESSION FOR THE POLYNOMIAL J_n , OR RATHER FOR THE INVERSE POLYNOMIAL \overline{J}_n GIVEN BY REVERSING THE COEFFICIENTS, I.E. ITS ROOTS ARE THE INVERSE SEMI-POSITIONS: $\overline{J}_M = 2 \left(1 + \cos \frac{6M\pi}{N} \right)$, 67
 $0 < M < N = \lfloor 6n \rfloor$, $n \equiv \pi \pmod{2}$. ALSO WE NORMALIZE \overline{J}_n SO THAT ITS CONSTANT TERM IS $(-1)^n$, I.E. $\overline{J}_n(0) = (-1)^n$. THESE FACTS, ALONG WITH THE DEGREE OF \overline{J}_n , WHICH IS

$$N' = \begin{cases} 3n & \text{if } n \geq 0 \\ -3n-1 & \text{if } n < 0 \end{cases}$$

COMPLETELY DETERMINE \overline{J}_n

WE FIRST SIMPLIFY THE DESCRIPTION OF THE ROOTS \overline{J}_M .

a) THE NUMBERS $\cos \frac{6M\pi}{N}$, $0 < M < N$, $n \equiv \pi \pmod{2}$ ARE THE SAME AS $\cos \frac{6M'\pi}{N}$, $0 < M' < \frac{N}{2}$, NO PARITY CONDITION: THIS IS BECAUSE IF $M > \frac{N}{2}$, THEN $M' = N - M$ IS $< \frac{N}{2}$ AND $\cos \frac{6M\pi}{N} = \cos \frac{6M'\pi}{N}$.

b) THE NUMBERS $\cos \frac{6M\pi}{N}$, $0 < M < \frac{N}{2}$ ARE THE SAME AS THE NUMBERS $\cos \frac{2M'\pi}{N}$, $0 < M' < \frac{N}{2}$. TO SEE THIS, NOTE THAT THERE IS A UNIQUE M' IN $[\frac{N}{2}, \frac{N}{2}]$ SUCH THAT $M' \equiv 3M \pmod{N}$, AND SO A UNIQUE M'' IN $[1, \frac{N}{2}]$ SUCH THAT $M' = \pm 3M \pmod{N}$. BUT $\cos \frac{6M\pi}{N}$ IS EQUAL TO $\cos \frac{2M''\pi}{N}$ IFF $6M \equiv \pm 2M'' \pmod{N}$, I.E. IFF $M' \equiv \pm 3M \pmod{N}$; THUS THE TWO SETS ARE EQUAL.

c) THE NUMBERS $\cos \frac{2M'\pi}{N}$, $0 < M' < \frac{N}{2}$ ARE THE SAME AS $\cos \frac{2M''\pi}{6N+1}$, $0 < M'' < \frac{N}{2}$, SINCE THE SIGN OF $6M\pi$ DOESN'T AFFECT THE VALUE OF THE COSINE.

d) THE RANGE $0 < M' < \frac{N}{2}$ IS THE SAME AS $0 < M'' \leq N'$

$$e) 2 \left(1 + \cos \frac{2M''\pi}{6N+1} \right) = 4 \cos^2 \frac{M''\pi}{N}$$

HENCE WE HAVE

PROP 3.3: \overline{J}_n IS THE UNIQUE POLYNOMIAL OF DEGREE N' WHOSE ROOTS ARE $\overline{J}_M = 4 \cos^2 \frac{M\pi}{6N+1}$, $0 < M \leq N'$, AND $\overline{J}_n(0) = (-1)^n$, WHERE $N' = \begin{cases} 3n & \text{if } n \geq 0 \\ -3n-1 & \text{if } n < 0 \end{cases}$.

IN THIS FORMULATION, WE CAN DESCRIBE \overline{J}_n MORE EASILY.

DEFINITION $X_n(t) = \frac{T_{6n+2} - T_{6n}}{2(t^2-1)}$: ITS DEGREE IS $2N'$ AND

ITS ROOTS ARE $u_m = \cos \frac{m\pi}{6n+1}$, $0 < m < |6n+1|$. NOTE THAT

68.

$u_{|6n+1-m} = \cos(\pi - \frac{m\pi}{6n+1}) = -u_m$, SO THE $2N'$ ROOTS OCCUR IN \pm PAIRS;

IN FACT, BY 7) ABOVE, T_{6n+2} AND T_{6n} ARE EVEN FUNCTIONS, I.E.

FUNCTIONS OF t^2 , AND SO THERE IS X_n . WE THEN REPLACE t^2 IN X_n

BY $\frac{s}{4}$ I.E. DEFINE $Y_n(s) = X_n(\frac{1}{2}\sqrt{s})$ AND HAVE:

a) DEG $Y_n = N'$

b) $Y_n(0) = (-1)^n$

c) THE ROOTS OF Y_n ARE $4u_m^2 = 4\cos^2 \frac{m\pi}{6n+1} = s_m$, $0 < m < N'$

WE ONLY NEED TO SHOW b): BUT $Y_n(0) = X_n(0) = \frac{T_{6n+2}(0) - T_{6n}(0)}{-2}$
 $= \frac{(-1)^{3n+1} - (-1)^{3n}}{-2} = (-1)^{3n} = (-1)^n$.

BUT WE HAVE ALREADY SEEN THAT \bar{S}_n IS THE UNIQUE POLYNOMIAL SATISFYING a), b), c), SO WE HAVE:

THM 3: THE INVERSE "SEINFUSION" POLYNOMIAL \bar{S}_n FOR $\frac{1}{n}$ SURGERY ON THE TREFOIL KNOT IS

$$\bar{S}_n(s) = 2 \frac{T_{6n+2}(\frac{1}{2}\sqrt{s}) - T_{6n}(\frac{1}{2}\sqrt{s})}{s-4}, \text{ WHERE}$$

T_n IS THE n^{TH} Tchebychev POLYNOMIAL,

PROOF: $\bar{S}_n(s) = X_n(\frac{1}{2}\sqrt{s})$ HAS DEGREE $2(t^2-1) = 2(\frac{s}{4}-1) = \frac{1}{2}(s-4)$.

G. WE CAN NOW EASILY PROVE THE RECURSION FORMULA FOR \bar{S}_n .

FIRST WE NEED A SLIGHTLY GENERALIZED RECURSION FOR TcheBYCHEV POLYNOMIALS:

PROP 39 $2T_m T_n = T_{m+n} + T_{m-n}$

PROOF: $2 \left(\frac{e^{ix} + e^{-ix}}{2} \right) \left(\frac{e^{iy} + e^{-iy}}{2} \right) = \frac{e^{(m+n)iy} + e^{-(m+n)iy}}{2} + \frac{e^{(m-n)iy} + e^{-(m-n)iy}}{2}$

$= T_{m+n} + T_{m-n}$

We are going to use Prop. 39 in the form $2T_6 T_n = T_{n+6} + T_{n-6}$.
 Applying this to $X_n = \frac{T_{n+2} - T_{n-2}}{2(t^2-1)}$ we get

69.

$$2T_6 X_n = \frac{(T_{n+8} + T_{n-4}) - (T_{n+2} + T_{n-8})}{2(t^2-1)} = X_{n+1} + X_{n-1},$$

i.e., $X_{n+1} = 2T_6 X_n - X_{n-1}$

Now substitute $t = \frac{1}{2}\sqrt{5}$ and get

$$\bar{J}_{n+1}(s) = 2T_6\left(\frac{1}{2}\sqrt{5}\right)\bar{J}_n(s) - \bar{J}_{n-1}$$

But from the table of Chebyshev polynomials,

$$2T_6\left(\frac{1}{2}\sqrt{5}\right) = 2\left(32\frac{s^3}{64} - 48\frac{s^2}{16} + 12\frac{s}{4} - 1\right) = s^3 - 6s^2 + 9s - 2$$

Hence:

Theorem 4: The inverse semi-torsion polynomial for $\frac{1}{2}\sqrt{5}$ surgery on the trefoil knot satisfies the recursive formula

$$\bar{J}_{-1} = -s^2 + 3s - 1$$

$$\bar{J}_0 = 1$$

$$\bar{J}_{n+1} = D \bar{J}_n - \bar{J}_{n-1} \text{ where } D = s^3 - 6s^2 + 9s - 2$$

H. WE GIVE HERE A GENERAL FORMULA FOR THE TORSION OF A KNOT COMPLEMENT K . LET A PROJECTION OF THE KNOT BE GIVEN, DETERMINING A WITTENBERG PRESENTATION $\{m = m_1, m_2, \dots, m_n \mid r_1, \dots, r_{n-1}\}$. 70. THE r_j 'S ARE THE CROSSING RELATIONS AND AS IS WELL KNOWN WE NEED ONLY $n-1$ OF THEM. THEN THE KNOT COMPLEMENT IS CONSTRUCTED FROM n 1-HANDLES $\{m_i\}$ AND $n-1$ 2-HANDLES $\{P_j\}$ ATTACHED TO r_j . AS SUCH WE CAN USE THE CHAIN COMPLEX OF $\mathbb{Z}[t]$ TO COMPUTE H_1 :

$$V_p \otimes (0 \rightarrow \overset{\text{GENS}}{dR} \xrightarrow{dr} \overset{\text{GENS}}{dF} \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0)$$

i.e. $0 \rightarrow V^{n-1} \xrightarrow{dr} V^n \xrightarrow{\epsilon} V \rightarrow 0$, WHERE $\epsilon = \begin{pmatrix} m_1-1 \\ m_2-1 \\ \vdots \\ m_n-1 \end{pmatrix}$ AND dr

IS THE $(2n-2) \times 2n$ MATRIX GIVEN BY $dr(dP_j) = dr_j = \sum_{i=1}^n \frac{\partial r_j}{\partial m_i} dm_i$

i.e.
$$dr = \begin{pmatrix} \frac{\partial r_1}{\partial m_1} & \frac{\partial r_1}{\partial m_2} & \frac{\partial r_1}{\partial m_3} & \dots \\ \frac{\partial r_2}{\partial m_1} & \dots & \dots & \dots \\ \vdots & & & \end{pmatrix}$$

SUPPOSE NOW THAT THE REP ρ SATISFIES $H_0(K, V_\rho) = 0$ (FOR EXAMPLE, IF ρ IS IRREDUCIBLE). THEN ϵ IS ONTO AND, IF e_1, e_2 IS A BASIS OF V , WE CAN CHOOSE LIFTS ~~...~~

$E_1 = (v_{11}, v_{12}, \dots, v_{1n})$, $E_2 = (v_{21}, v_{22}, \dots, v_{2n})$ OF e_1, e_2 TO V^n . DEFINE THE $2n \times 2n$ MATRIX M WHOSE FIRST $2n-2$ ROWS ARE dr AND LAST TWO ROWS ARE E_1, E_2 .

THEOREM 5: IF $H_0(K, V_\rho) = 0$ AND M IS CONSTRUCTED AS ABOVE, THEN $H_1(K, V_\rho) = 0$ IFF $\det M \neq 0$, AND IN THIS CASE, $\tau(K, V_\rho) = \det M$.

PROOF: NOTE FIRST THAT IF $H_1(K, V_\rho) = 0$, THEN $\tau = \det M$ BY DEFINITION: $\det M$ IS CLEARLY THE PRODUCT VOLUME IN V^n OF $\text{Im } dr$ AND THE LIFTED V .

TO SEE THAT $\det M \neq 0$ IMPLIES ACYCLICITY, NOTE THAT THE LAST TWO ROWS ARE COMPLEMENTARY TO $\text{Ker } \epsilon \supset \text{Im } dr$. HENCE $\det M \neq 0$ IFF $\text{RANK } dr = 2n-2$. BUT $\dim(\text{Ker } \epsilon) = 2n-2 = \dim V^{n-1}$, SO

|| RANK $dr = 2n-2$ iff $H_1(K, V_p) = H_2(K, V_p) = 0$, QED.

THEOREM 6: Let ρ be a representation of the knot group π such that $\det(\mu-1) \neq 0$, i.e. $\mu-1$ is nonsingular. Then $\det M$ (M as above) is $\frac{\det A}{\det(\mu-1)}$, where A is the $(n-2) \times (n-2)$ submatrix of dr :

$$\begin{pmatrix} \frac{\partial r_1}{\partial x_2} & \frac{\partial r_1}{\partial x_3} & \dots \\ \frac{\partial r_2}{\partial x_2} & \frac{\partial r_2}{\partial x_3} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

AND THIS NUMBER IS THE TORSION IF $\det A \neq 0$

PROOF: $\mu-1$ INVERTIBLE $\Rightarrow E$ IS ONTO $\Rightarrow H_0 = 0$ SO WE CAN APPLY THEOREM 5.

WE MAY THEN CHOOSE $E_1 = (\mu-1)^{-1}e_1, 0, 0, \dots$, $E_2 = (\mu-1)^{-1}e_2, 0, 0, \dots$

AND M TAKES THE FORM

$$\begin{pmatrix} \boxed{\text{stuff}} & A & \dots \\ (\mu-1)^{-1}e_1 & 0 & 0 & \dots \\ (\mu-1)^{-1}e_2 & 0 & 0 & \dots \end{pmatrix}$$

AND THE THEOREM FOLLOWS.

REMARK 1: FOR A 2-BRIDGE KNOT WE GET $D = \frac{\partial r}{\partial x_2}$, $T = \frac{\det \frac{\partial r}{\partial x_i}}{\det(\mu-1)}$

REMARK 2: NOTE THAT $\det A$ IS ENTIRELY ANALOGOUS TO A STANDARD METHOD OF CALCULATING THE ALEXANDER POLYNOMIAL. MILNOR HAS A PARALLEL RESULT FOR THE ALEXANDER POLYNOMIAL IN "A DUALITY THEOREM FOR REIDEMEISTER TORSION" ANNALS, V.76 #1, 1962, PP. 137-147

REMARK 3: CONCERNING THE CONDITION $\det(\mu-1) \neq 0$, SUPPOSE WE ARE LIVING ON A REPRESENTATION VARIETY \hat{R} OF THE KNOT GROUP. THEN IF THE CONDITION HOLDS AT ONE POINT, IT HOLDS IN A ZARISKI OPEN SET U AND $\frac{\det D}{\det(\mu-1)}$ IS AN ALGEBRAIC FUNCTION ON \hat{R} . IN U , THIS FUNCTION EQUALS $\det M$, SO THIS IS TRUE ON ALL OF \hat{R} , AND HENCE $T = \frac{\det D}{\det(\mu-1)}$ EVERYWHERE THE RIGHT SIDE IS DEFINED AND NON-ZERO.

REMARK 4: AS USUAL, THIS FORMULA IS EASY TO COMPUTE, GIVEN A REPRESENTATION, BUT FINDING THE REPRESENTATIONS MAY BE DIFFICULT.

TO THIS END, WE NEED THE CHEBYCHEV POLYNOMIALS DEFINED AS FOLLOWS:
 $\cos n\theta$ CAN CLEARLY BE WRITTEN AS A POLYNOMIAL IN $\sin\theta, \cos\theta$, AND IN FACT,
 SINCE $\sin^2 = 1 - \cos^2$, AS A POLY $T_n(\cos\theta) + \sin\theta \cdot U_n(\cos\theta)$ FOR SOME
 POLY T_n, U_n . BUT $\cos n\theta$ IS AN EVEN FUNCTION, SO $U_n = 0$. 78.
 LET $t = \cos\theta$; THEN $T_n(t)$ IS THE n TH CHEBYCHEV POLYNOMIAL.

IT HAS THE FOLLOWING PROPERTIES:

1) $T_0 = 1, T_1 = t$

2) $T_{-n} = T_n$, SINCE $\cos(-n\theta) = \cos n\theta$

3) $T_n(1) = T_n(\cos 0) = \cos(n \cdot 0) = 1$

4) $T_n(-1) = T_n(\cos \pi) = \cos n\pi = (-1)^n$

5) $T_n(0) = T_n(\cos \frac{\pi}{2}) = \cos \frac{n\pi}{2} = \begin{cases} 0 & \text{IF } n \text{ IS ODD} \\ (-1)^{n/2} & \text{IF } n \text{ IS EVEN} \end{cases}$

6) $T_{n+1} = 2tT_n - T_{n-1}$ (RECURSION FORMULA)

PROOF: $2tT_n = (e^{in\theta} + e^{-in\theta}) \frac{e^{in\theta} + e^{-in\theta}}{2} = \frac{e^{i(n+1)\theta} + e^{i(n-1)\theta}}{2} + \frac{e^{-i(n+1)\theta} + e^{-i(n-1)\theta}}{2} = T_{n+1} + T_{n-1}$

7) BY INDUCTION, USING 3) & 6), DEGREE $T_n = n$ AND T_n IS EVEN/ODD FUNCTION IF n IS EVEN/ODD.

HERE IS A SHORT LIST OF CHEBYCHEV POLYNOMIALS FOUND BY USING THE RECURSION FORMULA:

$T_0 = 1, T_1 = t, T_2 = 2t^2 - 1, T_3 = 4t^3 - 3t, T_4 = 8t^4 - 8t^2 + 1, T_5 = 16t^5 - 20t^3 + 5t, T_6 = 32t^6 - 48t^4 + 18t^2 - 1$

NOW LET $u_m = \cos \frac{m\pi}{6n+1}, 0 \leq m \leq |6n+1|$. WE HAVE:

$T_{6n+2}(u_m) = \cos \frac{(6n+2)m\pi}{6n+1} = \cos \left(m\pi + \frac{m\pi}{6n+1} \right) = (-1)^m u_m$

$T_{6n}(u_m) = \cos \frac{6nm\pi}{6n+1} = \cos \left(4\pi - \frac{m\pi}{6n+1} \right) = (-1)^m u_m$

SO $T_{6n+2} - T_{6n}$ HAS ALL THE u_m AS ITS ROOTS. SINCE ITS DEGREE IS THE SAME AS THE NUMBER OF u_m 's, I.E. $\begin{cases} 6n+2 \text{ IF } n \geq 0 \\ -6n \text{ IF } n < 0 \end{cases} = 2N+2$ AND THE u_m ARE DISTINCT, THEY ARE EXACTLY THE ROOTS OF $T_{6n+2} - T_{6n}$.
 NOW $u_0 = 1$ AND $u_{|6n+1|} = -1$, SO $T_{6n+2} - T_{6n}$ IS DIVISIBLE BY $t^2 - 1$.