GENERALIZATION OF POINCARÉ'S FORMULA IN THE THEORY OF PROBABILITY

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Let $p_{[m]}(1, \cdots, n)$, $(0 \leq m \leq n)$ denote the probability of the occurrence of exactly m events among the n arbitrary events $E_1, \cdots, E_n$; and $p_m(1, \cdots, n)$ $(1 \leq m \leq n)$ that of at least m. Let $p_{\nu_1 \cdots \nu_i}$ $(1 \leq i \leq n)$, where $(\nu_1 \cdots \nu_i)$ is a combination (without repetition) out of $(1, \cdots, n)$, denote the probability of the occurrence of $E_{\nu_1}, \cdots, E_{\nu_i}$ (without regard to the other events); and

$$S_0 = 1, \qquad S_i = \sum_{(\nu_1 \cdots \nu_i)} p_{\nu_1 \cdots \nu_i},$$

where the summation extends to all the combinations with i members out of $(1, \cdots, n)$.

Then Poincaré's formula may be written as follows:

$$p_{[0]}(1, \cdots, n) = \sum_{i=0}^{n} (-1)^i S_i.$$

An equivalent formula is:

$$p_1(1, \cdots, n) = \sum_{i=1}^{n} (-1)^{i-1} S_i.$$

The following conventions concerning the binomial coefficients are made:

$$\binom{0}{0} = 1, \qquad \binom{a}{b} = 0 \qquad \text{if } a < b \text{ or } b < 0.$$

Two generalizations, possibly due to de Mises, are

$$p_{[m]}(1, \cdots, n) = \sum_{i=m}^{n} (-1)^{(i-m)} \binom{i}{m} S_i;$$

$$p_m(1, \cdots, n) = \sum_{i=m}^{n} (-1)^{(i-m)} \binom{i-1}{m-1} S_i.$$

We notice that the probabilities appearing on the left-hand sides of these formulas are symmetrical with respect to the set of suffixes $(1, \cdots, n)$, and the sums on the right-hand sides are symmetrical in the same way.

As a natural generalization let us consider a probability which is symmetrical with respect to certain sub-sets of $(1, \cdots, n)$. We divide the n events into r sets:

$$E_{\nu_{11}}, \cdots, E_{\nu_{1n_1}}; E_{\nu_{21}}, \cdots, E_{\nu_{2n_2}}; \cdots; E_{\nu_{r1}}, \cdots, E_{\nu_{rn_r}};$$

where $n_1 + n_2 + \cdots + n_r = n$. And we ask for the probability that out of the first set of n_1 events exactly m_1 events occur; and out of the second set of n_2 events exactly m_2 events occur; and so on; and finally, out of the rth set of n_r events exactly m_r events occur. When this problem is solved the analogous problem in which we replace some of the words "exactly" by "at least" can also be solved.

We denote the required probability by the left-hand side of the following *generalized Poincaré's formula*:

$$p_{[m_1],[m_2],\cdots,[m_r]}(\nu_{11}\cdots\nu_{1n_1};\nu_{21}\cdots\nu_{2n_2};\cdots;\nu_{r1}\cdots\nu_{rn_r})$$

$$= \sum_{i_1=m_1}^{n_1}\sum_{i_2=m_2}^{n_2}\cdots\sum_{i_r=m_r}^{n_r}(-1)^{i_1+i_2+\cdots+i-m_1-m_2-\cdots-m_r} \binom{i_1}{m_1}\binom{i_2}{m_2}\cdots\binom{i_r}{m_r}S_{i_1,i_2,\cdots,i_r}, \tag{1}$$

where

$$S_{i_1,i_2,\cdots,i_r} = \Sigma p_{\alpha_{11}\cdots\alpha_{1i_1}\alpha_{21}\cdots\alpha_{2i_2}\cdots\alpha_{r1}\cdots\alpha_{ri_r}},$$

the summation extending to all those combinations of α's such that for every $k = 1, \cdots, r$, $(\alpha_{k1}\cdots\alpha_{ki_k})$ is a combination of i_k members out of $(\nu_{k1}\cdots\nu_{kn_k})$.

PROOF: Let $p_{[\nu_1\cdots\nu_i]}$ denote the probability of the occurrence of the events $E_{\nu_1}, \cdots, E_{\nu_i}$ and *these only* out of $E_1, \cdots, E_n$. It is well-known and also easily seen that

$$p_{\alpha_1\cdots\alpha_a} = \sum_{b=0}^{n-a}\sum_{\beta} p_{[\alpha_1\cdots\alpha_a\beta_1\cdots\beta_b]}$$

where for a fixed b the second summation extends to all the combinations $(\beta_1\cdots\beta_b)$ of b members out of the "difference set" $(1, \cdots, n) - (\alpha_1\cdots\alpha_a)$.

Now let each p in each S on the right-hand side of (1) be decomposed into a sum of the $p_{[\nu_1\cdots\nu_i]}$'s in the last-written way. Consider a fixed

$$p_{[\mu_{11}\cdots\mu_{1j_1}\mu_{21}\cdots\mu_{2j_2}\cdots\mu_{r1}\cdots\mu_{rj_r}]},$$

where for every $k = 1, \cdots, r$, $(\mu_k, \cdots, \mu_{kj_k})$ is a combination of j_k members out of $(\nu_{k1}\cdots\nu_{kn_k})$. It appears once in exactly $\binom{j_1}{i_1}\binom{j_2}{i_2}\cdots\binom{j_r}{i_r}$ terms in $S_{i_1,i_2,\cdots i_r}$. Hence, its total contribution to the right-hand side of (1) is

$$\sum_{i_1=m_1}^{n_1}\sum_{i_2=m_2}^{n_2}\cdots\sum_{i_r=m_r}^{n_r}(-1)^{i_1+i_2+\cdots+i_r-m_1-m_2-\cdots-m_r} \cdot\binom{i_1}{m_1}\binom{i_2}{m_2}\cdots\binom{i_r}{m_r}\binom{j_1}{i_1}\binom{j_2}{i_2}\cdots\binom{j_r}{i_r}$$

$$= \prod_{k=1}^{r}\binom{m_k}{j_k}\sum_{i_k=m_k}^{n_k}(-1)^{i_k-m_k}\binom{j_k-m_k}{i_k-m_k}$$

$$= \prod_{k=1}^{r}\binom{j_k}{m_k}\sum_{h_k=0}^{j_k-m_k}(-1)^{h_k}\binom{j_k-m_k}{h_k}$$

$$= \prod_{k=1}^{r}\binom{j_k}{m_k}\cdot\begin{cases}1 & \text{if } j_k = m_k\\ 0 & \text{if otherwise}\end{cases}$$

$$= \begin{cases}1 & \text{if } j_k = m_k \text{ for every } k = 1, \cdots, r\\ 0 & \text{otherwise.}\end{cases}$$

Therefore after the decompositions and the collecting of terms, the only p's remaining on the right-hand side of (1) are those in which for every $k = 1, \cdots, j$ we have $j_k = m_k$. Thus the right-hand side is reduced to

$$\Sigma p_{[\mu_{11} \cdots \mu_{1m_1} \mu_{21} \cdots \mu_{2m_2} \cdots \mu_{r1} \cdots \mu_{rm_r}]},$$

where the summation extends to all those combinations of μ's such that for every $k = 1, \cdots, r$, $(\mu_{k1} \cdots \mu_{km_k})$ is a combination of m_k members out of $(\nu_{k1} \cdots \nu_{kn_k})$. This is clearly equal to the left-hand side of (1). Q. E. D.

If we replace "exactly m_k" by "at least m_k" in the definition of the probability just considered, we replace in our notation the square-bracketed $[m_k]$ by an unbracketed m_k and we replace in our formula $\binom{i_k}{m_k}$ by $\binom{i_k - 1}{m_k - 1}$. This is proved as before, noting that we have

$$\sum_{i_k = m_k}^{n_k} (-1)^{i_k - m_k} \binom{i_k - 1}{m_k - 1}\binom{j_k}{i_k} = 1 \qquad \text{for } j_k = m_k, \cdots, n_k;$$

and identity which can be proved by induction on j_k.

A parallel generalization of Poincaré's formula is as follows: We ask for the probability that *either* out of the first set exactly m_1 events occur; *or* out of the second exactly m_2; $\cdots$; *or* finally, out of the rth set exactly m_r. That is, instead of repeated conjunctions we may consider repeated disjunctions. We denote the required probability by the left hand side of (2), then it is given in terms of the p's defined above in (1) by the right-hand side below:

$$(2) \quad W_{[m_1],[m_2],\cdots,[m_r]}(\nu_{11}, \cdots, \nu_{1n_1}; \nu_{21} \cdots \nu_{2n_2}; \cdots; \nu_{r1} \cdots \nu_{rn_r})$$
$$= p_{m_1,m_2,\cdots,m_r} - p_{m_1+1,m_2+1,\cdots,m_r+1}.$$

Other events symmetrical with respect to each of the sub-sets, in whose definition the words "and", "or", "exactly", "at least" appear arbitrarily, may be considered.

Lastly, we only mention that as a first application the formula (1) can be used to establish the formula

$$(n - k)\Sigma p_m(\nu_1 \cdots \nu_k) = (k + 1 - m)\Sigma p_m(\nu_1 \cdots \nu_{k+1}) + m\Sigma p_{m+1}(\nu_1 \cdots \nu_{k+1}),$$

first obtained by P. L. Hsu. For its significance we may refer to [1], and a continuation of that paper to be published shortly.

REFERENCE

[1] K. L. Chung, "On the probability of the occurrence of least m events among n arbitrary events," *Annals of Math. Stat.*, Vol. 12 (September 1941).