FURTHER RESULTS ON PROBABILITIES OF A FINITE NUMBER OF EVENTS

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In a recent paper[1] the author has generalized some inequalities of Fréchet to the following:

Let $n \geqq a \geqq m \geqq 1$, and let

$$\binom{n-m}{a-m}^{-1} P_a^{(m)}((\nu)) = A_a^{(m)}.$$

$$\Delta F(a) = F(a) - F(a+1), \qquad \Delta^h F(a) = \Delta(\Delta^{h-1} F(a));$$

then

$$\Delta A_a^{(m)} \geqq 0, \qquad \Delta^2 A_a^{(m)} \geqq 0.$$

Using a generalized Poincaré's formula, P. L. Hsu has improved these inequalities to the recurrence formula stated below.

Hsu's formula is

$$\Delta A_a^{(m)} = \frac{m}{n-m} A_{a+1}^{(m+1)}. \tag{1}$$

Proof: We have

$$p_m((\alpha)) = \sum_{b=m}^{a} (-1)^{b-m} \binom{b-1}{m-1} S_b((\alpha)).$$

For a fixed "a" summing over all $(\alpha) \epsilon (\nu)$,

$$\sum_{(\alpha)\epsilon(\nu)} p_m((\alpha)) = \sum_{b=m}^{a} (-1)^{b-m} \binom{b-1}{m-1}\binom{n-b}{a-b} S_b((\nu))$$

$$A_a^{(m)} = \binom{n-1}{m-1} \sum_{b=m}^{a} (-1)^{b-m} \binom{a-m}{b-m}\binom{n-1}{b-1}^{-1} S_b((\nu))$$

$$\Delta A_a^{(m)} = \binom{n-1}{m-1}\left\{\sum_{b=m}^{a} (-1)^{b-m}\left[\binom{a-m}{b-m} - \binom{a+1-m}{b-m}\right]\binom{n-1}{b-1}^{-1} S_b((\nu)) - (-1)^{a+1-m} \cdot \binom{n-1}{a}^{-1} S_{a+1}((\nu))\right\}$$

$$= \binom{n-1}{m-1} \sum_{b=m+1}^{a+1} (-1)^{b-m-1} \binom{a-m}{b-m-1}\binom{n-1}{b-1}^{-1} S_b((\nu))$$

$$= \frac{m}{n-m} A_{a+1}^{(m+1)}, \qquad \text{Q.E.D.}$$

[1] "On the probability of the occurrence of at least m events among n arbitrary events," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 328–338. We use throughout the same notation used in this paper, and that referred to in footnote 3.

Applying the formula repeatedly, we obtain for $0 \leqq h \leqq n - a$,

$$\Delta^h A_a^{(m)} = \binom{a+m-1}{h}\binom{n-m}{h}^{-1} A_{a+h}^{(m+h)}.$$

Since every $A \geqq 0$, we have, for $0 \leqq h \leqq n - a$,

$$\Delta^h A_a^{(m)} \geqq 0,$$

which includes my former results.

Further, we may write (1) as

$$(n-a)P_a^{(m)} = (a+1-m)P_{a+1}^{(m)} + mP_{a+1}^{(m+1)} \tag{2}$$

or

$$(a+1)P_{a+1}^{(m)} - (n-a)P_a^{(m)} = m\,(P_{a+1}^{(m)} - P_{a+1}^{(m+1)}) = mP_{a+1}^{[m]}$$

It follows that

$$(a+1)P_{a+1}^{(m)} - (n-a)P_a^{(m)} \geqq 0. \tag{3}$$

From (2) it also follows that

$$(n-a)P_a^{(m)} - (a+1-m)P_{a+1}^{(m)} \geqq 0, \tag{4}$$

which is the same as $\Delta A_a^{(m)} \geqq 0$. Combining (3) and (4) we obtain

$$\frac{n-a}{a+1} P_a^{(m)} \leqq P_{a+1}^{(m)} \leqq \frac{n-a}{a+1-m} P_a^{(m)}.$$

If we take the special case $m = 1$ and instead of the original events $E_1, \cdots, E_n$ consider their negations, we easily obtain

$$\frac{n-a}{a+1}\left\{\binom{n}{a} - S_a((\nu))\right\} \leqq \binom{n}{a} - S_{a+1}((\nu)) \leqq \frac{n-a}{a}\left\{\binom{n}{a} - S_a((\nu))\right\}.$$

This is equivalent to a result given by Fréchet[2].

There is an analogue of Hsu's formula for $P_{[m]}$, as follows:
Let $n \geqq a \geqq m \geqq 1$, and let

$$\binom{n-m}{a-m}^{-1} P_a^{[m]} = B_a^{[m]},$$

then

$$\Delta B_a^{[m]} = \frac{m+1}{n-m} B_{a+1}^{[m+1]}.$$

It follows that for $0 \leqq h \leqq n - a$,

$$\Delta^h B_a^{[m]} = \binom{m+h}{m}\binom{n-m}{h}^{-1} B_{a+h}^{[m+h]};$$

$$\Delta^h B_a^{[m]} \geqq 0.$$

[2] "Evénements compatibles et probabilités fictives," *C. R. Acad. Sc.*, Vol. 208 (1939).

The other results on p_m in the paper[1] also have analogues for $p_{[m]}$. For the result on conditions of existence see the author's recent paper[3]. Here we shall state the following extension of Boole's inequality.

For $2l + 1 \leqq n - a$ and $2l \leqq n - a$ respectively, we have

$$\sum_{i=0}^{2l+1} (-1)^i \binom{m+i}{m} S_{m+i}((\nu)) \leqq p_{[m]}((\nu)) \leqq \sum_{i=0}^{2l} (-1)^i \binom{m+i}{m} S_{m+i}((\nu)).$$

PROOF: We have

$$S_{m+i}((\nu)) = \sum_{h=0}^{n=m} \binom{m-h}{m+i} p_{[m+h]}((\nu)).$$

Hence,

$$\begin{aligned} \sum_{i=0}^{g} (-1)^i \binom{m+i}{m} S_{m+i}((\nu)) &= \sum_{h=0}^{n-m} \left\{ \sum_{i=0}^{g} (-1)^i \binom{m+i}{m} \binom{m+h}{m+i} \right\} p_{[m+h]}((\nu)) \\ &= p_{[m]}((\nu)) + \sum_{h=1}^{n-m} \binom{m+h}{m} \sum_{i=0}^{g} (-1)^i \binom{h}{i} p_{[m+h]}((\nu)) \\ &= p_{[m]}((\nu)) + \sum_{h=1}^{n-m} \binom{m+h}{m} (-1)^g \binom{h-1}{g} p_{[m+h]}((\nu)). \end{aligned}$$

The inequalities follow immediately.

Finally, we record two formulas which express $p_a((\nu))$ in terms of $P_b^{(m)}((\nu))$ and in terms of $P_b^{[m]}((\nu))$ for a fixed m and ranging b's. Formulas which express $P_{[a]}((\nu))$ in both ways have been given[2].

We have,

$$\binom{c-1}{m-1} p((\gamma)) = \sum_{b=m}^{c} (-1)^{b-m} \sum_{(\beta) \epsilon (\gamma)} p_m((\beta))$$

Hence

$$\begin{aligned} \binom{c-1}{m-1} S_c((\nu)) &= \sum_{(\gamma) \epsilon (\nu)} \sum_{b=m}^{c} (-1)^{b-m} \sum_{(\beta) \epsilon (\gamma)} p_m((\beta)) \\ &= \sum_{b=m}^{c} (-1)^{b-m} \binom{n-b}{c-b} \sum_{(\beta) \epsilon (\nu)} p_m((\beta)) \end{aligned}$$

$$S_c((\nu)) = \binom{c-1}{m-1}^{-1} \sum_{b=m}^{c} (-1)^{b-m} \binom{n-b}{c-b} P_b^{(m)}$$

By a generalized Poincaré's formula, we get

$$\begin{aligned} p_a((\nu)) &= \sum_{b=m}^{n} (-1)^{b-m} \sum_{c=\max(a,b)}^{n} (-1)^{c-a} \binom{c-1}{a-1} \binom{n-b}{c-b} \binom{c-1}{m-1}^{-1} P_b^{(m)} \\ &= \sum_{b=m+n-a}^{n} (-1)^{n-a+b-m} \binom{b-m}{n-a} \binom{a-1}{m-1}^{-1} P_b^{(m)}. \end{aligned}$$

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[3] "On fundamental systems of probabilities of a finite number of events," *Annals of Math. Stat.*, Vol. 14 (1943), pp. 123–134.

Similarly we have

$$S_c((\nu)) = \binom{c}{m}^{-1} \sum_{b=m}^{c} (-1)^{b-m} \binom{n-b}{c-b} P_b^{[m]}$$

$$p_a((\nu)) = \sum_{b=m}^{n} (-1)^{b-m} \left\{ \sum_{c=\max(a,b)}^{n} (-1)^{c-a} \binom{c-1}{a-1} \binom{n-b}{c-b} \binom{c}{m}^{-1} \right\} P_b^{[m]}$$

It remains to be seen whether the series in the curl brackets can be summed.

Using a formula in footnote 3, we may obtain the desired formula in another way. We have, in fact,

$$\begin{aligned} p_a((\nu)) &= \sum_{c=a}^{m} p_{[c]}((\nu)) \\ &= \sum_{c=a}^{n} \sum_{b=m+n-c}^{n} (-1)^{n-c+b-m} \binom{b-m}{n-c} \binom{c}{m}^{-1} P_b^{[m]}((\nu)) \\ &= \sum_{b=m}^{m+n-a} (-1)^{b-m} \left\{ \sum_{c=m+n-b}^{n} (-1)^{n-c} \binom{b-m}{n-c} \binom{c}{m}^{-1} \right\} P_b^{[m]}((\nu)) \\ &\quad + \sum_{b=m+n-a+1}^{n} (-1)^{b-m} \left\{ \sum_{c=a}^{n} (-1)^{n-c} \binom{b-m}{n-c} \binom{c}{m}^{-1} \right\} P_b^{[m]}((\nu)). \end{aligned}$$

The "complete" series

$$\begin{aligned} \sum_{c=m+n-b}^{n} (-1)^{n-c} \binom{b-m}{n-c} \binom{c}{m}^{-1} &= \sum_{d=0}^{b-m} (-1)^{d} \binom{b-m}{d} \binom{n-d}{m}^{-1} \\ &= (-1)^{b-m} \frac{m}{n} \binom{n-1}{b-1}^{-1}. \end{aligned}$$

The "incomplete" series we denote by

$$K(n, a, b, m) = \sum_{c=a}^{n} (-1)^{n-c} \binom{b-m}{n-c} \binom{c}{m}^{-1} = \sum_{d=0}^{n-a} (-1)^{d} \binom{b-m}{d} \binom{n-d}{m}^{-1}.$$

Then we may write

$$p_a((\nu)) = \sum_{b=m}^{m+n-a} \frac{m}{n} \binom{n-1}{b-1}^{-1} P_b^{[m]} + \sum_{b=m+n-a+1}^{n} (-1)^{b-m} K(n, a, b, m) P_b^{[m]}.$$