FURTHER RESULTS ON PROBABILITIES OF A FINITE NUMBER OF EVENTS

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In a recent paper\(^1\) the author has generalized some inequalities of Fréchet to the following:

Let \( n \geq a \geq m \geq 1 \), and let

\[
\left( \frac{n - m}{a - m} \right)^{-1} P_a^{(m)}((\nu)) = A_a^{(m)}.
\]

\[
\Delta F(a) = F(a) - F(a + 1), \quad \Delta^k F(a) = \Delta(\Delta^{k-1} F(a));
\]

then

\[
\Delta A_a^{(m)} \geq 0, \quad \Delta^2 A_a^{(m)} \geq 0.
\]

Using a generalized Poincaré's formula, P. L. Hsu has improved these inequalities to the recurrence formula stated below.

Hsu's formula is

\[
(1) \quad \Delta A_a^{(m)} = \frac{m}{n - m} A_{a+1}^{(m+1)}.
\]

PROOF: We have

\[
p_m((\alpha)) = \sum_{b=m}^{a} (-1)^{b-m} \binom{b - 1}{m - 1} S_b((\alpha)).
\]

For a fixed \("a"\) summing over all \((\alpha) \in (\nu),\)

\[
\sum_{(\alpha) \in (\nu)} p_m((\alpha)) = \sum_{b=m}^{a} (-1)^{b-m} \binom{b - 1}{m - 1} \binom{n - b}{a - b} S_b((\nu))
\]

\[
A_a^{(m)} = \binom{n - 1}{m - 1} \sum_{b=m}^{a} (-1)^{b-m} \binom{a - m}{b - m} \binom{n - 1}{b - 1} S_b((\nu))
\]

\[
\Delta A_a^{(m)} = \binom{n - 1}{m - 1} \left\{ \sum_{b=m}^{a} (-1)^{b-m} \left[ \binom{a - m}{b - m} - \binom{a - 1}{b - 1} S_b((\nu)) \right] \right\}
\]

\[
\frac{n - 1}{a} S_{a+1}((\nu)))
\]

\[
= \binom{n - 1}{m - 1} \sum_{b=m+1}^{a+1} (-1)^{b-m-1} \binom{a - m}{b - m - 1} \binom{n - 1}{b - 1} S_b((\nu))
\]

\[
= \frac{m}{n - m} A_{a+1}^{(m+1)}, \quad \text{Q.E.D.}
\]

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\(^1\) On the probability of the occurrence of at least \(m\) events among \(n\) arbitrary events,” Annals of Math. Stat., Vol. 12 (1941), pp. 328-338. We use throughout the same notation used in this paper, and that referred to in footnote 3.

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Applying the formula repeatedly, we obtain for $0 \leq h \leq n - a$,

$$\Delta^h A^{(m)}_a = \left( \frac{a + m - 1}{h} \right) \left( \frac{n - m}{h} \right)^{h-1} A^{(m+h)}_{a+h}.$$  

Since every $A \geq 0$, we have, for $0 \leq h \leq n - a$,

$$\Delta^h A^{(m)}_a \geq 0,$$

which includes my former results.

Further, we may write (1) as

$$\frac{(n - a)P^{(m)}_a - (a + 1 - m)P^{(m)}_{a+1}}{(a + 1 - m)P^{(m)}_{a+1}} = m \left( \frac{P^{(m)}_a - P^{(m+1)}_{a+1}}{P^{(m+1)}_{a+1}} \right) = mP^{(m)}_{a+1}$$

It follows that

$$\frac{(a + 1)P^{(m)}_{a+1} - (n - a)P^{(m)}_a}{(a + 1)P^{(m)}_{a+1}} \geq 0.$$  

From (2) it also follows that

$$\frac{(n - a)P^{(m)}_a - (a + 1 - m)P^{(m)}_{a+1}}{(a + 1 - m)P^{(m)}_{a+1}} \geq 0,$$

which is the same as $\Delta A^{(m)}_a \geq 0$. Combining (3) and (4) we obtain

$$\frac{n - a}{a + 1} P^{(m)}_a \leq P^{(m)}_{a+1} \leq \frac{n - a}{a + 1} P^{(m)}_a.$$  

If we take the special case $n_a = 1$ and instead of the original events $E_1, \ldots, E_n$ consider their negations, we easily obtain

$$\frac{n - a}{a + 1} \left\{ \left( \frac{n}{a} \right) - S_a((\nu)) \right\} \leq \frac{n}{a} \left( \frac{n}{a} - S_{a+1}(\nu) \right) \leq \frac{n - a}{a} \left\{ \left( \frac{n}{a} \right) - S_a((\nu)) \right\}.$$  

This is equivalent to a result given by Fréchet.

There is an analogue of Hsu's formula for $P^{(m)}_{[m]}$, as follows:

Let $n \geq a \geq m \geq 1$, and let

$$\left( \frac{n - m}{a - m} \right)^{m} P^{(m)}_a = B^{[m]}_a,$$

then

$$\Delta B^{[m]}_a = \frac{m + 1}{n - m} B^{[m+1]}_{a+1}.$$  

It follows that for $0 \leq h \leq n - a$,

$$\Delta^h B^{[m]}_a = \left( \frac{m + h}{m} \right) \left( \frac{n - m}{h} \right)^{h-1} B^{[m+h]}_{a+h}$$

$$\Delta^h B^{[m]}_a \geq 0.$$  

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\*\*\*Evénements compatibles et probabilités fictives,* C. R. Acad. Sc., Vol. 208 (1939).\**
The other results on \( p_a \) in the paper\(^1\) also have analogues for \( p_{[m]} \). For the result on conditions of existence see the author’s recent paper\(^2\). Here we shall state the following extension of Boole’s inequality.

For \( 2l + 1 \leq n - a \) and \( 2l \leq n - a \) respectively, we have

\[
\sum_{i=0}^{2l+1} (-1)^i \binom{m+i}{m} S_{m+i}((\nu)) \leq p_{[m]}((\nu)) \leq \sum_{i=0}^{2l} (-1)^i \binom{m+i}{m} S_{m+i}((\nu)).
\]

**Proof:** We have

\[
S_{m+i}((\nu)) = \sum_{k=0}^{n-m} \binom{m-h}{m+i} p_{[m+h]}((\nu)).
\]

Hence,

\[
\sum_{i=0}^{2l} (-1)^i \binom{m+i}{m} S_{m+i}((\nu)) = \sum_{k=0}^{n-m} \left( \sum_{i=0}^{2l} (-1)^i \binom{m+i}{m} \binom{m+h}{m+i} \right) p_{[m+h]}((\nu))
\]

\[
= p_{[m]}((\nu)) + \sum_{h=1}^{n-m} \binom{m+h}{m} \sum_{i=0}^{2l} (-1)^i \binom{h}{i} p_{[m+h]}((\nu))
\]

\[
= p_{[m]}((\nu)) + \sum_{h=1}^{n-m} \binom{m+h}{m} (-1)^h \binom{h-1}{g} p_{[m+h]}((\nu)).
\]

The inequalities follow immediately.

Finally, we record two formulas which express \( p_a((\nu)) \) in terms of \( P_b^{(m)}((\nu)) \) and in terms of \( P_b^{[m]}((\nu)) \) for a fixed \( m \) and ranging \( b \)'s. Formulas which express \( P_{[a]}((\nu)) \) in both ways have been given\(^3\).

We have,

\[
\binom{c-1}{m-1} p((\gamma)) = \sum_{b=m}^{c} (-1)^{b-m} \sum_{(\beta) \in (\gamma)} p_m((\beta))
\]

Hence

\[
\binom{c-1}{m-1} S_a((\nu)) = \sum_{(\gamma) \in (\nu)} \sum_{b=m}^{c} (-1)^{b-m} \sum_{(\beta) \in (\gamma)} p_m((\beta))
\]

\[
= \sum_{b=m}^{c} (-1)^{b-m} \binom{n-b}{c-b} \sum_{(\beta) \in (\nu)} p_m((\beta))
\]

\[
S_a((\nu)) = \binom{c-1}{m-1}^{-1} \sum_{b=m}^{c} (-1)^{b-m} \binom{n-b}{c-b} P_b^{(m)}
\]

By a generalized Poincaré's formula, we get

\[
p_a((\nu)) = \sum_{b=m}^{n-a} (-1)^{b-m} \sum_{c=\max(a,b)}^{n-a} (-1)^{c-a} \binom{c-1}{a-1} \binom{n-b}{c-b} \binom{c-1}{m-1}^{-1} P_b^{(m)}
\]

\[
= \sum_{b=m+n-a}^{n} (-1)^{n-a+b-m} \binom{b-m}{n-a} \binom{a-1}{m-1}^{-1} P_b^{(m)}.
\]

Similarly we have

\[ S_c(n) = \binom{c}{m}^{-1} \sum_{b=m}^{n} (-1)^{b-m} \binom{n-b}{c-b} P_b^{[m]} \]

\[ p_a(n) = \sum_{b=m}^{n} (-1)^{b-m} \left\{ \sum_{c=\max(a,b)}^{n} (-1)^{c-a} \binom{c-1}{a-1} \binom{n-b}{c-b} \binom{c-1}{m} \right\} P_b^{[m]} \]

It remains to be seen whether the series in the curl brackets can be summed.

Using a formula in footnote 3, we may obtain the desired formula in another way. We have, in fact,

\[ p_a(n) = \sum_{c=a}^{n} p_{[a]}(n) \]

\[ = \sum_{c=a}^{n} \sum_{b=m+n-c}^{n} (-1)^{n-c+b-m} \binom{b-m}{n-c} \binom{c}{m}^{-1} P_b^{[m]}(n) \]

\[ = \sum_{b=m}^{n} (-1)^{b-m} \left\{ \sum_{c=m+n-b}^{n} (-1)^{n-c} \binom{b-m}{n-c} \binom{c}{m}^{-1} \right\} P_b^{[m]}(n) \]

\[ + \sum_{b=m+n-a+1}^{n} (-1)^{b-m} \left\{ \sum_{c=a}^{n} (-1)^{n-c} \binom{b-m}{n-c} \binom{c}{m}^{-1} \right\} P_b^{[m]}(n). \]

The "complete" series

\[ \sum_{c=m+n-b}^{n} (-1)^{n-c} \binom{b-m}{n-c} \binom{c}{m}^{-1} = \sum_{d=0}^{b-m} (-1)^{d} \binom{b-m}{d} \binom{n-d}{m}^{-1} \]

\[ = (-1)^{b-m} \frac{m}{n} \frac{n-1}{b-1}. \]

The "incomplete" series we denote by

\[ K(n, a, b, m) = \sum_{c=a}^{n} (-1)^{n-c} \binom{b-m}{n-c} \binom{c}{m}^{-1} = \sum_{d=0}^{n-a} (-1)^{d} \binom{b-m}{d} \binom{n-d}{m}^{-1}. \]

Then we may write

\[ p_a(n) = \sum_{b=m}^{n} \frac{m}{n} \binom{n-1}{b-1}^{-1} P_b^{[m]} + \sum_{b=m+n-a+1}^{n} (-1)^{b-m} K(n, a, b, m) P_b^{[m]}. \]