THE APPROXIMATE DISTRIBUTION OF STUDENT'S STATISTIC

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Summary. It is well known that various statistics of a large sample (of size n) are approximately distributed according to the normal law. The asymptotic expansion of the distribution of the statistic in a series of powers of $n^{-\frac{1}{2}}$ with a remainder term gives the accuracy of the approximation. H. Cramér [1] first obtained the asymptotic expansion of the mean, and recently P. L. Hsu [2] has obtained that of the variance of a sample. In the present paper we extend the Cramér-Hsu method to Student's statistic. The theorem proved states essentially that if the population distribution is non-singular and if the existence of a sufficient number of moments is assumed, then an asymptotic expansion can be obtained with the appropriate remainder. The first four terms of the expansion are exhibited in formula (35).

1. In a fundamental paper[1] P. L. Hsu [2] has devised a method for obtaining the asymptotic expansion of the distribution of various statistics. The present paper deals with the so-called Student statistic.

Let

$$\xi_1, \xi_2, \cdots, \xi_n$$

be n independent random variables having the same probability distribution represented by a distribution function $P(x)$. The rth moment and rth absolute moment are denoted by α_r and β_r respectively. It is assumed that $\alpha_1 = 0$ and that for a certain $k \geqq 3$, $\beta_k < \infty$ and that $\alpha_2 > 0$. Hence there is no loss of generality in assuming that $\alpha_2 = 1$.

Student's statistic is defined as

$$\bar{\xi}\left(\frac{\sum_{r=1}^{n}(\xi_r - \bar{\xi})^2}{n(n-1)}\right)^{-\frac{1}{2}} \quad \text{where} \quad \bar{\xi} = \frac{1}{n}\sum_{r=1}^{n}\xi_r .$$

For brevity, we consider

$$n\bar{\xi}\left(\sum_{r=1}^{n}(\xi_r - \bar{\xi})^2\right)^{-\frac{1}{2}}.$$

Let its distribution function be denoted by $F(z)$, i.e.,

$$Pr\left\{n\bar{\xi}\left(\sum_{r=1}^{n}(\xi_r - \bar{\xi})^2\right)^{-\frac{1}{2}} \leqq z\right\} = F(z).$$

[1] The definitions of the various constants A, A_k, Q_k, Λ_k, ϑ, Θ, Θ_k, are the same as in Hsu's paper.

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Discarding the case $k = 3$ where we can prove a more precise result and the singular case which can be shown to admit no asymptotic expansion in the sense of Cramér [1], we shall prove in this paper the following theorem:

THEOREM. *Let $P(x)$ be non-singular and $\alpha_{2k} < \infty$ for some integer $k \geqq 4$. Then*

$$(1) \qquad F(z) = \Phi(z) + \chi(z) + R(z), \qquad \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{1}{2}y^2}\, dy,$$

where $\chi(z)$ is a linear combination of the derivatives $\Phi'(z), \cdots, \Phi^{(3k-10)}(z)$ with each coefficient of the form $n^{-\frac{1}{2}\nu}(1 \leqq \nu \leqq k-3)$ times a quantity depending only on α_3, $\cdots, \alpha_{k-1}$ whose beginning terms are given in (35) *and where*

$$(2) \qquad |R(z)| \leqq Q_k(1 + |z|^{2k-3})n^{-\alpha_0}, \qquad \alpha_0 = \frac{(k-1)\left[\dfrac{k}{2}\right]}{2\left(\left[\dfrac{k}{2}\right] + 1\right)},$$

where Q_k is a constant depending on k and $P(x)$.[1]

We shall need some of Hsu's lemmas, i.e., his lemma 3, lemma 7 (both for the particular case $m = 2$) and lemma 8. These we shall quote with this numbering. The application of Hsu's method to Student's statistic depends on the following lemma.

2. LEMMA A. *For $u \geqq -1$, $l \geqq 1$, we have*

$$1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - j\right)\Gamma(j+1)} u^j - \left(1 + \sum_{j=1}^{2l-1} \frac{(-1)^j\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - j\right)\Gamma(j+1)}\right) u^{2l}$$

$$\leqq \sqrt{1+u} \leqq 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - j\right)\Gamma(j+1)} u^j.$$

PROOF. By Taylor's expansion of $\sqrt{1+u}$, we have

$$\sqrt{1+u} = 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - j\right)\Gamma(j+1)} u^j$$

$$+ \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - 2l\right)\Gamma(2l+1)} (1 + \vartheta u)^{-\frac{1}{2}(4l-1)} u^{2l},$$

whence it follows that $(1 + \vartheta u)^{-\frac{1}{2}(4l-1)}$ is finite, and positive. The right-hand side inequality follows.

Similarly, if $u \geqq 0$,

$$\sqrt{1+u} \geqq 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j + \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-2l\right)\Gamma(2l+1)} u^{2l}$$

$$\geqq 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j - \left(1 + \sum_{j=1}^{2l-1} \frac{(-1)^j\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)}\right)u^{2l}$$

since by a well-known result on the binomial theorem we have

$$1 + \sum_{j=1}^{\infty} \frac{(-1)^j\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} = \sqrt{1-1} = 0.$$

For $-1 \leqq u < 0$, we have

$$1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j - \sqrt{1+u} = \frac{N}{D}, \quad \text{say.}$$

For $-1 \leqq u < 0$, we have

$$D = 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j + \sqrt{1+u}$$

$$\geqq 1 + \sum_{j=1}^{2l-1} \frac{(-1)^j\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)}.$$

Next,

$$N = \left(1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j\right)^2 - (1+u)$$

is a polynomial in u of the form

$$u^{2l}(a_0 + a_1u + \cdots + a_{2l}u^{2l})$$

where $a_0 > 0$ and the successive coefficients have alternating signs; hence for $-1 \leq u \leqq 0$, $a_0 + a_1u + \cdots + a_{2l}u^{2l}$ assumes its maximum at $u = -1$. This maximum is obtained by putting $u = -1$ in the numerator, hence for $-1 \leqq u < 0$,

$$N \leqq u^{2l}\left(1 + \sum_{j=1}^{2l-1} \frac{(-1)^j \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - j\right)\Gamma(j+1)}\right)^2$$

The left-hand side inequality in the lemma now follows.

For brevity we write the inequalities as

$$(3) \quad 1 + P_{2l}(u) = 1 + P_{2l-1}(u) - b_{2l}u^{2l} \leqq \sqrt{1+u} \leqq 1 + P_{2l-1}(u), \quad b_{2l} > 0.$$

3. We write

$$\sum_{r=1}^{n} (\xi_r - \bar{\xi})^2 = \sum_{r=1}^{n} \xi_r^2 - n\bar{\xi}^2 = n + \sqrt{n(\alpha_4 - 1)}\, X - Y^2,$$

where

$$X = \sum_{r=1}^{n} \frac{\xi_r^2 - 1}{\sqrt{n(\alpha_4 - 1)}}, \quad Y = \sqrt{n}\bar{\xi}.$$

Then Student's statistic may be written as

$$n\bar{\xi}\left(\sum_{r=1}^{n} (\xi_r - \bar{\xi})^2\right)^{-\frac{1}{2}} = Y\left(1 + \sqrt{\frac{\alpha_4 - 1}{n}}\, X - \frac{Y^2}{n}\right)^{-\frac{1}{2}}.$$

Then, for every z, we have

$$F(z) = Pr\left\{Y\left(1 + \sqrt{\frac{\alpha_4 - 1}{n}}\, X - Y^2\right)^{-\frac{1}{2}} \leqq z\right\}$$
$$= Pr\left\{\sqrt{1 + \frac{z^2}{n}}\, Y \leqq z\sqrt{1 + \sqrt{\frac{\alpha_4 - 1}{n}}\, X}\right\}.$$

For brevity let

$$\sqrt{1 + \frac{z^2}{n}}\, Y = V, \qquad \sqrt{\frac{\alpha_4 - 1}{n}}\, X = U.$$

Suppose $z \leqq 0$; then we have by (3),

$$(4) \quad \begin{gathered} z + zP_{2l-1}(U) \leqq z\sqrt{1+U} \leqq z + zP_{2l}(U) \\ Pr\{V \leqq z + zP_{2l-1}(U)\} \leqq F(z) \leqq Pr\{V \leqq z + zP_{2l}(U)\} \end{gathered}$$

Suppose $z > 0$; then we have by Lemma A a similar inequality with the extreme terms exchanged.

Now we take $l = \left[\frac{k}{2}\right]$, and fix it henceforth.

Our next step is to obtain an asymptotic expansion for

$$Pr\{V \leqq z + zP_m(U)\} = Pr\left\{Y \leq z\left(1+\frac{z^2}{n}\right)^{-\frac{1}{2}} + z\left(1+\frac{z^2}{n}\right)^{-\frac{1}{2}} P_m\left(\sqrt{\frac{\alpha_4 - 1}{n}}\, X\right)\right\}$$

with $m = 2l - 1$ or $2l$, $l \geqq 1$.

Let b be any real number, and

$$z\left(1+\frac{z^2}{n}\right)^{-\frac{1}{2}} P_m\left(\sqrt{\frac{\alpha_4 - 1}{n}}\, X\right) = L_m(X).$$

Until section 12, we shall write simply $L(x)$ for either of the $L_m(x)$.

4. Let W be the probability function of the distribution of the random point (X, Y) and let $f(t_1, t_2)$ be the characteristic function.

$$W(S) = Pr\{(X, Y) \in S\} \quad \text{for every Borel set } S \text{ in } \mathfrak{R}_2$$

$$f(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it_1x + it_2y}\, dW = \left\{p\left(\frac{t_1}{\sqrt{n}}, \frac{t_2}{\sqrt{n}}\right)\right\}^n$$

$$p(t_1, t_2) = \int_{-\infty}^{\infty} e^{it_1(\alpha_4-1)^{-\frac{1}{2}}(x^2-1)+it_2x}\, dP.$$

Then

$$(5) \qquad Pr\{Y \leqq b + L(X)\} = \iint\limits_{y \leqq b + L(x)} dW = \iint\limits_{y \leqq b} dW + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y)\, dW$$

where

$$G(x, y) = \begin{cases} 1 & \text{if } b < y \leqq b + L(x), \\ -1 & \text{if } b + L(x) < y \leqq b, \\ 0 & \text{otherwise} \end{cases}$$

We approximate $G(x, y)$ by $H(x, y)$, where

$$H(x, y) = \begin{cases} e^{-\epsilon x^{2l}} & \text{if } b < y \leqq b + L(x) \\ -e^{-\epsilon x^{2l}} & \text{if } b + L(x) < y \leqq b \\ 0 & \text{otherwise} \end{cases}$$

We approximate dW by $(w(x, y) + \gamma(x, y))\, dx\, dy$, where

$$w(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x - it_2y}\phi(t_1, t_2)\, dt_1\, dt_2$$

$$\gamma(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x - it_2y}\phi(t_1, t_2)\psi(it_1, it_2)\, dt_1\, dt_2$$

$$\phi(t_1, t_2) = e^{-\frac{1}{2}(t_1^2 + 2\rho t_1t_2 + t_2^2)}, \; \rho = \mathrm{E}\left(\frac{(\xi^2 - 1)\xi}{\sqrt{\alpha_4 - 1}}\right) = \frac{\alpha_3}{\sqrt{\alpha_4 - 1}}$$

and $\psi(it_1, it_2)$ is given in Lemma 3 by taking therein $\zeta_1 = \frac{\xi^2 - 1}{\sqrt{\alpha_4 - 1}}$, $\zeta_2 = \xi$, ξ being any of the ξ_i's.

We write

$$
(6)\quad
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y-u)\,dW \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G(x,y-u) - H(x,y-u))\,dW \\
&- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G(x,y-u) - H(x,y-u))(w(x,y) + \gamma(x,y))\,dy\,dx \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y-u)\,dW \\
&- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y-u)(w(x,y) + \gamma(x,y))\,dy\,dx \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y-u)(w(x,y) + \gamma(x,y))\,dy\,dx
\end{aligned}
$$

5. We have

$$(7)\quad | G(x, y-u) - H(x, y-u) | \leqq 1 - e^{-\epsilon x^{2l}} \leqq \epsilon x^{2l}$$

$$(8)\quad \left| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G(x,y-u) - H(x,y-u))\,dW \right| \leqq \int_{-\infty}^{\infty} \epsilon x^{2l}\,dW = \epsilon E(X^{2l}) \leqq Q_k\epsilon$$

since

$$E(X^{2l}) \leqq A_k E\left(\frac{\xi^2 - 1}{\sqrt{\alpha_4 - 1}}\right)^{2l} \leqq Q_k$$

where Q_k depends on $\alpha_3, \cdots, \alpha_{2k}$.

Similarly,

$$(9)\quad \left| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G(x,y-u) - H(x,y-u))(w(x,y) + \gamma(x,y))\,dy\,dx \right| \leqq Q_k\epsilon$$

Next,

$$
(10)\quad
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y-u)(w(x,y) + \gamma(x,y))\,dy\,dx \\
&= \int\int_{y \leqq u+b+L(x)} (w(x,y) + \gamma(x,y))\,dy\,dx - \int\int_{y \leqq u+b} (w(x,y) + \gamma(x,y))\,dy\,dx
\end{aligned}
$$

where the first term on the right-hand side, regarded as a function of $n^{-\frac{1}{2}}$, has a Taylor expansion in powers of $n^{-\frac{1}{2}}$, whose first few terms we shall compute in section 9; for the present let us denote it by $B(u + b) + C(u + b)n^{-\frac{1}{2}(k-2)}$ where $C = C(u + b)$ is a constant depending on k, $P(x)$ and z, a more explicit estimate of which will be given in section 10.

Further, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y - u)\, dW = \int\int_{y \leqq u+b+L(x)} dW - \int\int_{y \leqq u+b} dW \tag{11}$$

by Cramér's asymptotic expansion for the mean $\sqrt{n}Y$, and as is also shown in Hsu's paper we have

$$\int\int_{y \leqq u+b} dW - \int\int_{y \leqq u+b} (w(x,y) + \gamma(x,y))\, dy\, dx = \Lambda_k n^{-\frac{1}{2}(k-2)} \tag{12}$$

Collecting all the results from (5)–(12), we get

$$\begin{aligned}\int\int_{y \leqq u+b+L(x)} dW &- B(u + b) - C(u + b)n^{-\frac{1}{2}(k-2)} \\ &= \Lambda_k(\epsilon + n^{-\frac{1}{2}(k-2)}) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y - u)\, dW \\ &- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y - u)(w(x,y) + \gamma(x,y))\, dy\, dx\end{aligned}$$

Now we use A. C. Berry's weighting factor $\dfrac{1 - \cos Tu}{u^2}$ and obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2}\Big(\int\int_{y \leqq u+b+L(x)} dW &- B(u + b) - C(u + b)\Big) du \\ &= \Lambda_k T(\epsilon + n^{-\frac{1}{2}(k-2)}) \\ &+ \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y - u)\, dW \\ &- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y - u)(w(x,y) + \gamma(x,y))\, dy\, dx\Big) du\end{aligned} \tag{13}$$

since

$$\int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2}\, du = \pi T.$$

6. To transform the triple integral on the right-hand side of (13) we use the Fourier transform as Hsu did.

Let

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x - it_2y} H(x, y)\, dy\, dx = h(t_1, t_2);$$

then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}H(x,y-u)\,dy\,dx = e^{-it_2u}h(t_1,t_2)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}\left(\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}H(x,y-u)\,du\right)dy\,dx = \begin{cases}\pi(T-|t_2|)h(t_1,t_2) & \text{if } |t_2|\le T\\ 0 & \text{otherwise}\end{cases}$$

since

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}e^{-it_2u}\,du = \begin{cases}\pi(T-|t_2|) & \text{if } |t_2|\le T\\ 0, & \text{otherwise}\end{cases}$$

By Fourier inversion we have, almost everywhere,

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}H(x,y-u)\,du = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-T}^{T}e^{it_1x+it_2y}(T-|t_2|)h(t_1,t_2)\,dt_2\,dt_1$$

Hence

$$\text{(14)}\quad \begin{aligned}&\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(x,y-u)\,dW\,du\\ &\qquad = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-T}^{T}(T-|t_2|)f(t_1,t_2)h(t_1,t_2)\,dt_2\,dt_1\end{aligned}$$

Similarly we obtain

$$\text{(15)}\quad \begin{aligned}&\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(x,y-u)(w(x,y)+\gamma(x,y))\,dy\,dx\right)du\\ &\qquad = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-T}^{T}(T-|t_2|)\phi(t_1,t_2)\{1+\psi(it_1,it_2)\}h(t_1,t_2)\,dt_2\,dt_1\end{aligned}$$

From (14) and (15) we obtain

$$\text{(16)}\quad \begin{aligned}&\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(x,y-u)\,dW\right.\\ &\qquad\left.-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(x,y-u)(w(x,y)+\gamma(x,y))\,dy\,dx\right)du\\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-T}^{T}(T-|t_2|)\{f(t_1,t_2)\\ &\qquad -\phi(t_1,t_2)(1+\psi(it_1,it_2))\}h(t_1,t_2)\,dt_2\,dt_1.\end{aligned}$$

7. To estimate the double integral on the right-hand side of (16) we break it up into parts and use the following estimates of $h(t_1, t_2)$.

LEMMA B. *We have for* $l = \left[\frac{k}{2}\right] \geqq 1$,

$$(1) \qquad | h(t_1, t_2) | \leqq A_k | z | \sum_{j=1}^{2l} (\alpha_4 - 1)^{\frac{1}{2}j} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l};$$

$$(2) \qquad | h(t_1, t_2) | \leqq Q_k t_1^{-2} z^2 N(| t_2 |, n^{-1/2}, \epsilon^{-1/2l})$$

where $N(| t_2 |, n^{-1/2}, \epsilon^{-1/2l})$ *is a polynomial with constant coefficients in the indicated arguments.*

PROOF.

$$| h(t_1, t_2) | = \iint_{b < y \leqq b + L(x)} e^{-it_1x - it_2y - \epsilon x^{2l}} \, dy \, dx - \iint_{b + L(x) < y \leqq b} e^{-it_1x - it_2y - \epsilon x^{2l}} \, dy \, dx$$

$$= \left(\int_{L(x) \geqq 0} \int_b^{b+L(x)} - \int_{L(x) < 0} \int_{b+L(x)}^{b} \right) e^{-it_1x - it_2y - \epsilon x^{2l}} \, dy \, dx$$

$$= (-it_2)^{-1} e^{-it_2b} \int_{-\infty}^{\infty} e^{-it_1x - \epsilon x^{2l}} (e^{it_2L(x)} - 1) \, dx.$$

Hence

$$| h(t_1, t_2) | \leqq | t_2 |^{-1} \int_{-\infty}^{\infty} | t_2 L(x) | e^{-\epsilon x^{2l}} \, dx.$$

Since

$$| L(x) | \leqq A_k | z | \sum_{j=1}^{2l} (\alpha_4 - 1)^{\frac{1}{2}j} n^{-\frac{1}{2}j} | x |^j$$

we obtain

$$| h(t_1, t_2) | \leqq A_k | z | \sum_{j=1}^{2l} (\alpha_4 - 1)^{\frac{1}{2}j} n^{-\frac{1}{2}j} \int_{-\infty}^{\infty} | x |^j e^{-\epsilon x^{2l}} \, dx$$

$$\leqq A_k | z | \sum_{j=1}^{2l} (\alpha_4 - 1)^{\frac{1}{2}j} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l}.$$

Next, we write

$$h(t_1, t_2) = (-it_2)^{-1} e^{-it_2b} \int_{-\infty}^{\infty} u''(x) v(x) \, dx$$

with

$$u''(x) = e^{-it_1x}, \qquad v(x) = e^{-\epsilon x^{2l}} (e^{-it_2L(x)} - 1).$$

Integrating by parts twice, we get

$$h(t_1, t_2) = (-it_2)^{-1} e^{-it_2b} \int_{-\infty}^{\infty} u(x) v''(x) \, dx$$

whence

$$| h(t_1, t_2) | \leqq A_k t_1^{-2} \int_{-\infty}^{\infty} e^{-\epsilon x^{2l}} \{ | L''(x) | + \epsilon x^{2l-1} | L'(x) | + \epsilon | x |^{2l-2} | L(x) |$$

$$+ \epsilon^2 x^{2l-2} | L(x) | + | t_2 || L'^2(x) | \} dx \leqq Q_k t_1^{-2} (| z | + z^2) N(| t_2 |, n^{-\frac{1}{2}}, \epsilon^{-\frac{1}{2}l})$$

The lemma is proved.

Now we write

$$(17) \quad \int_{-\infty}^{\infty} \int_{-T}^{T} (T - | t_2 |) \{ f - \phi(1 + \psi) \} h \, dt_2 \, dt_1 = \iint_{\substack{|t_1| \leqq Q_k n^{\frac{1}{2}} \\ |t_2| \leqq Q_k n^{\frac{1}{2}}}} + \iint_{\substack{|t_1| > Q_k n^{\frac{1}{2}} \\ |t_2| \leqq T}} + \iint_{\substack{|t_1| \leqq Q_k n^{\frac{1}{2}} \\ Q_k n^{\frac{1}{2}} < |t_2| \leqq T}} = I_1 + I_2 + I_3 .$$

On I_1 we use Lemma 3 and Lemma B, (1):

$$| I_1 | \leqq Q_k | z | \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T \left(\sum_{j=1}^{l} n^{-\frac{1}{2}j} \epsilon^{-\frac{1}{2}(k-2)} \right) n^{-\frac{1}{2}(k-2)}$$

$$\cdot \left\{ \sum_{i=1}^{2} (| t_i |^k + \cdots + | t_i |^{3(k-2)} \right\} e^{-\frac{1}{4}(1-\rho^2)(t_1^2+t_2^2)} \, dt_2 \, dt_1$$

$$\leqq Q_k | z | T n^{-\frac{1}{2}(k-2)} \sum_{j=1}^{2l} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l}.$$

On I_2 we use Lemma 7 and Lemma B (2). Since $| t_1 | > Q_k n^{\frac{1}{2}}$, $| \Phi(1 + \psi) | \leqq e^{-nQ_k}$, and by Lemma 7, $p(t_1 n^{-\frac{1}{2}}, t_2 n^{-\frac{1}{2}}) = e^{-Q_k}$ so that $| f(t_1, t_2) | \leqq e^{-nQ_k}$, $| f - \Phi(1 + \psi) | \leqq e^{-nQ_k}$

$$I_2 \leqq Q_k z^2 \iint_{\substack{|t_1| > Q_k n^{\frac{1}{2}} \\ |t_2| \leqq T}} T t_1^{-2} e^{-nQ_k} N(| t_2 |, n^{-\frac{1}{2}}, \epsilon^{-1/2l}) \, dt_2 \, dt_1 .$$

Let $\epsilon = n^{-\beta}$, $\beta > 0$, then it is evident that

$$| I_2 | \leqq Q_k z^2.$$

Similarly using Lemma 7 and Lemma B, (1) on I_3 we see that

$$| I_3 | \leqq Q_k | z | .$$

Therefore

$$(18) \quad \left| \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-T}^{T} (T - | t_2 |) \{ f(t_1, t_2) - \phi(t_1, t_2)(1 + \psi(it_1, it_2)) \} h(t_1, t_2) \, dt_2 \, dt_1 \right| \leqq Q_k \left(| z | + z^2 + | z | T n^{-\frac{1}{2}(k-2)} \sum_{j=1}^{2l} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l} \right).$$

8. Combining (13), (16), (17) we obtain

$$(19)\quad \begin{aligned} &\left| \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \left(\iint_{y \leqq u+b+L(x)} dW - B(u + b) - C(u + b)n^{-\frac{1}{2}(k-2)} \right) du \right| \\ &\leqq Q_k \left(T\epsilon + Tn^{-\frac{1}{2}(k-2)} + |z| + z^2 + |z| \, Tn^{-\frac{1}{2}(k-2)} \sum_{j=1}^{2l} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l} \right). \end{aligned}$$

Now we shall choose T and ϵ suitably. Let

$$T = n^{\alpha}, \qquad \epsilon = n^{-\beta}, \qquad \alpha > 0, \qquad \beta > 0.$$

To make the right-hand side of (19) a constant depending on z only, we must have $\alpha \leqq \frac{1}{2}(k - 2) \cdot \beta \leqq \alpha$. Then

$$\sum_{j=1}^{2l} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l} = \sum_{j=1}^{2l} n^{((\beta - l)j + \beta)/2l}.$$

We must choose $\beta < k/2$, then

$$\sum_{j=1}^{2l} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l} \leqq A_k n^{(2\beta - l)/2l}.$$

To make the exponent as small as possible we choose $\beta = \alpha$, then

$$|z| \, Tn^{-\frac{1}{2}(k-2)} \sum_{j=1}^{2l} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l} \leqq A_k |z| \, n^{\alpha - \frac{1}{2}(k-2) + (2\alpha - l)/2l} = A_k |z| \, n^{(l+1)\alpha/l - \frac{1}{2}(k-1)}$$

since α is to be as large as possible, we choose

$$\alpha = \alpha_0 = \min\left(\frac{k-2}{2}, \frac{(k-1)l}{2(l+1)} \right), \qquad l = \left[\frac{k}{2} \right].$$

Then we obtain

$$(20)\quad \left| \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \left(\iint_{y \leqq u+b+L(x)} dW - B(u + b) - C(u + b)n^{-\frac{1}{2}(k-2)} \right) du \right| \leqq Q_k(1 + z^2).$$

Let $F^*(u)$ be the distribution function of $Y - L(X)$, and let

$$F_1(u) = B(u) + C(u)n^{-\frac{1}{2}(k-2)}$$

Then we may write (20) as

$$(21)\quad \left| \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{F^*(u + b) - F_1(u + b)\} \, du \right| \leqq Q_k(1 + z^2).$$

By the definition of $F_1(u)$ we see that the conditions in Lemma 8 are all satisfied with a certain constant D depending on k, $P(x)$, and z for the M therein. Then choosing b to be the a in Lemma 8, we obtain from Lemma 8 and (21),

$$(22)\quad DT\delta \left\{ 3 \int_0^{T\delta} \frac{1 - \cos x}{x^2} \, dx - \pi \right\} \leqq Q_k(1 + z^2)$$

where

$$\delta = \frac{1}{2D} \text{l.u.b.} \, | F^*(u) - F_1(u) | .$$

Now there exists A such that if $T\delta > A$, then

$$3 \int_0^{T\delta} \frac{1 - \cos x}{x^2} \, dx - \pi > 1,$$

hence it follows from (22) that

$$T\delta \leqq \max (A, D^{-1}Q_k(1 + z^2)).$$

Thus for another Q_k' exceeding both A and the above Q_k, we have

$$T\delta \leqq Q_k'(1 + z^2)$$

and so finally, dropping the prime,

(23) $$| F^*(u) - F_1(u) | \leqq Q_k(1 + z^2)DT^{-1} = Q_k(1 + z^2)Dn^{-\alpha_0}.$$

In particular, taking b to be $z(1 + n^{-1}z^2)^{-\frac{1}{2}} = z'$, say:

(24) $$Pr \{ Y - L(X) \leqq z' \} = B(z') + C(z')n^{-\frac{1}{2}(k-2)} + \Lambda_k(1 + z^2)Dn^{-\alpha}$$

where

$B(z') + C(z')n^{-\frac{1}{2}(k-2)}$ = the Taylor expansion with a remainder of

$$\iint_{y - L(x) \leqq z'} (w(x, y) + \gamma(x, y)) \, dy \, dx$$

and D is an upper bound for

$$| B'(u) + C'(u)n^{-\frac{1}{2}(k-2)} |.$$

9. Let $\lambda = n^{-\frac{1}{2}}$, and rewrite the $z' + L_{2l-1}(x)$, $l \geqq 2$ there as $g(\lambda)$:

$$g(\lambda) = z' \left(1 + \frac{(\alpha_4 - 1)^{1/2}}{2} \lambda x - \frac{\alpha_4 - 1}{8} \lambda^2 x^2 + \frac{(\alpha_4 - 1)^{3/2}}{16} \lambda^3 x^3 + \cdots \right)$$

Then

$$g(0) = z'$$

$$g'(0) = \frac{(\alpha_4 - 1)^{\frac{1}{2}}}{2} z'x$$

$$g''(0) = -\frac{\alpha_4 - 1}{4} z'x^2$$

$$g'''(0) = \frac{3(\alpha_4 - 1)^{3/2}}{8} z'x^3.$$

Let $p \geqq 0,\ q \geqq 0;\ w_{pq}(x, y) = \dfrac{\partial^{p+q}}{\partial x^p \partial y^q} w(x, y)$ where $w(x, y)$ is defined in section 4 and we know that

$$w(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-(x^2-2\rho xy+y^2)/2(1-\rho^2)}.$$

Let

$$f_{pq}(\lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} w_{pq}(x, y)\, dy\, dx. \tag{26}$$

Then

$$\begin{aligned}
f'_{pq}(\lambda) &= \int_{-\infty}^{\infty} g'(\lambda) w_{pq}(x, g(\lambda))\, dx \\
f''_{pq}(\lambda) &= \int_{-\infty}^{\infty} (g''(\lambda) w_{pq}(x, g(\lambda)) + g'^2(\lambda) w_{p,q+1}(x, g(\lambda)))\, dx \\
f'''_{pq}(\lambda) &= \int_{-\infty}^{\infty} (g'''(\lambda) w_{pq}(x, g(\lambda)) + 3g''(\lambda) g'(\lambda) w_{p,q+1}(x, g(\lambda)) \\
&\qquad + g'^3(\lambda) w_{p,q+2}(x, g(\lambda)))\, dx
\end{aligned} \tag{27}$$

Let

$$\begin{aligned}
\Phi = \Phi(z') &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z'} e^{\frac{1}{2}y^2}\, dy, \quad \Phi^{(q)} = \frac{d^q}{dz^q}\Phi(z)\Big|_{z=z'} \\
I^r_{pq} &= \int_{-\infty}^{\infty} x^r w_{pq}(x, z')\, dx
\end{aligned} \tag{28}$$

We have computed the following table of values of I^r_{pq}:

$r \backslash p$	0	1	2	3	$\geqq 4$
0	$\Phi^{(q)}$	0	0	0	0
1	$-\rho\Phi^{(q+1)}$	$-\Phi^{(q)}$	0	0	0
2	$\Phi^{(q)} + \rho^2\Phi^{(q+2)}$	$2\rho\Phi^{(q+1)}$	$2\Phi^{(q)}$	0	0
3	$-3\rho\Phi^{(q+1)} - \rho^3\Phi^{(q+3)}$	$-3\Phi^{(q)} - 3\rho^2\Phi^{(q+2)}$	$-6\rho\Phi^{(q+1)}$	$-6\Phi^{(q)}$	0

Next, we find, from (25)–(28),

$$\begin{aligned}
f_{00}(0) &= \Phi; \\
f_{pq}(0) &= I^0_{p,q-1} \text{ for } q \geqq 1; \\
f'_{pq}(0) &= \frac{(\alpha_4 - 1)^{\frac{1}{2}}}{4} z' I^1_{pq} \\
f''_{pq}(0) &= -\frac{\alpha_4 - 1}{4} z' I^2_{pq} + \frac{\alpha_4 - 1}{4} z'^2 I^2_{p,q+1} \\
f'''_{pq}(0) &= \frac{3(\alpha_4 - 1)^{3/2}}{8} z' I^3_{pq} - \frac{3(\alpha_4 - 1)^{3/2}}{8} z'^2 I^3_{p,q+1} + \frac{(\alpha_4 - 1)^{3/2}}{8} z'^3 I^3_{p,q+2}
\end{aligned} \tag{92}$$

Now we can expand

$$\iint_{y \leqq b+L(x)} w(x,y)\,dy\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} w(x,y)\,dy\,dx = f_{00}(\lambda)$$

Write the Taylor's series for $f_{00}(\lambda)$:

$$f_{00}(\lambda) = f_{00}(0) + f_{00}'(0)\lambda + \frac{f_{00}''(0)}{2}\lambda^2 + \frac{f_{00}'''(0)}{6}\lambda^3 + \cdots$$

Substituting from (29), we get

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} w(x,y)\,dy\,dx &= \Phi - \frac{(\alpha_4-1)^{1/2}}{2n^{1/2}}\rho z'\Phi^{(1)} \\
(30)\qquad &+ \frac{\alpha_4-1}{8n}\{-z'(\Phi^{(0)} + \rho^2\Phi^{(2)}) + z'^2(\Phi^{(1)} + \rho^2\Phi^{(3)})\} \\
&+ \frac{(\alpha_4-1)^{3/2}}{48n^{3/2}}\{3z'(-3\rho\Phi^{(1)} - \rho^3\Phi^{(3)}) - 3z'^2(-3\rho\Phi^{(2)} - \rho^3\Phi^{(4)}) \\
&\qquad + z'^3(-3\rho\Phi^{(3)} - \rho^3\Phi^{(5)})\} + \cdots
\end{aligned}$$

Further, we must obtain the beginning terms of $\gamma(x, y)$ as given in Lemma 3, for which purpose we refer to Hsu's paper. We have, in fact

$$\psi(it_1, it_2) = -\frac{iU^3}{6n^{1/2}} + \left(\frac{U_4}{24} - \frac{U_3^2}{36}\right)\frac{1}{n} + \left(\frac{U_5}{120} - \frac{U_3U_4}{72} + \frac{U_3^2}{216}\right)\frac{i}{n^{3/2}} + \cdots$$

where

$$\begin{aligned}
U_3 &= E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^3 \\
&= \alpha_3 t_2^3 + 3\sqrt{\alpha_4-1}\,t_2^2 t_1 + 3\,\frac{\alpha_5-2\alpha_3}{\alpha_4-1}t_2t_1^2 + \frac{\alpha_6-3\alpha_4+2}{(\alpha_4-1)^{3/2}}t_1^3 \\
U_4 &= E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^4 - 3\left(E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)\right)^2 \\
&= (\alpha_4-3)t_2^4 + 4\,\frac{\alpha_5-4\alpha_3}{\sqrt{\alpha_4-1}}t_2^3t_1 + \cdots \\
U_5 &= E\left((t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^5 - 10E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^2 E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^3 \\
&= (\alpha_5 - 10\alpha_3)t_2^5 + \cdots
\end{aligned}$$

To avoid the exhibition of very long expressions, let us separate the terms in $\psi(it_1, it_2)$ according to the powers of $n^{-1/2}$, and denote the terms of the power $n^{-1/2}\,n^{-1}, n^{-3/2}$ by ψ_1, ψ_2, ψ_3, respectively.

Thus $\psi_1 = -iU_3/6n^{1/2}$, and the corresponding $\gamma(x, y)$ is

$$\gamma_1(x,y) = -\frac{1}{6n^{1/2}}\left(\alpha_3 w_{03}(x,y) + 3\sqrt{\alpha_4 - 1}\, w_{12}(x,y) + 3\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1}\, w_{21}(x,y) + \cdots\right) \tag{31}$$

where, as hereafter, the terms omitted will yield nothing in the long run.

Now we have by (31) and (26),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} \gamma_1(x, y)\, dy\, dx$$

$$= -\frac{1}{6n^{1/2}}\left(\alpha_3 f_{03}(\lambda) + 3\sqrt{\alpha_4 - 1} f_{12}(x, y) + 3\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} f_{21}(x, y) + \cdots\right)$$

$$= -\frac{1}{6n^{1/2}}\left(\alpha_3 f_{03}(0) + 3\sqrt{\alpha_4 - 1} f_{12}(0) + 3\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} f_{21}(0) + \cdots\right)$$

$$-\frac{1}{6n}\left(\alpha_3 f'_{03}(0) + 3\sqrt{\alpha_4 - 1} f'_{12}(0) + 3\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} f'_{21}(0) + \cdots\right)$$

$$-\frac{1}{12n^{3/2}}\left(\alpha_3 f''_{03}(0) + 3\sqrt{\alpha_4 - 1} f''_{12}(0) + 3\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} f''_{21}(0) + \cdots\right)$$

$$= -\frac{1}{6n^{1/2}}(\alpha_3 l^0_{02}) - \frac{(\alpha_4 - 1)^{1/2}}{12n} z'(\alpha_3 l^1_{03} + 3\sqrt{\alpha_4 - 1}\, l^1_{12}) \tag{32}$$

$$+ \frac{\alpha_4 - 1}{48n^{3/2}}\left\{z'\left(\alpha_3 l^2_{03} + 3\sqrt{\alpha_4 - 1}\, l^2_{12} + 3\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1}\, l^2_{21}\right)\right.$$

$$\left. - z'^2\left(\alpha_3 l^2_{04} + 3\sqrt{\alpha_4 - 1}\, l^2_{13} + 3\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1}\, l^2_{22}\right)\right\} + \cdots$$

$$= -\frac{\alpha_3}{6n^{1/2}}\Phi^{(2)} + \frac{(\alpha_4 - 1)^{1/2}}{12n} z'(\alpha_3\rho\Phi^{(4)} + 3\sqrt{\alpha_4 - 1}\,\Phi^{(2)})$$

$$+ \frac{\alpha_4 - 1}{48n^{3/2}}\left\{z'\left[\alpha_3(\Phi^{(3)} + \rho^2\Phi^{(5)}) + 6\sqrt{\alpha_4 - 1}\,\rho\Phi^{(3)} + 6\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1}\,\Phi^{(1)}\right]\right.$$

$$\left. - z'^2\left[\alpha_3(\Phi^{(4)} + \rho^2\Phi^{(6)}) + 6\sqrt{\alpha_4 - 1}\,\rho\Phi^{(4)} + 6\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1}\,\Phi^{(2)}\right]\right\} + \cdots.$$

Similarly, omitting the intermediate steps to save space, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} \gamma_2(x, y)\, dy\, dx = \frac{1}{72n}\{3(\alpha_4 - 3)\Phi^{(3)} + 2\alpha_3^2\Phi^{(5)}\}$$

$$- \frac{(\alpha_4 - 1)^{1/2}}{144n^{3/2}} z'\left\{3(\alpha_4 - 3)\rho\Phi^{(5)} + 12\,\frac{\alpha_5 - 4\alpha_3}{\sqrt{\alpha_4 - 1}}\,\Phi^{(3)} + 2\alpha_3^2\rho\Phi^{(7)} + 12\alpha_3\sqrt{\alpha_4 - 1}\,\Phi^{(5)}\right\} + \cdots. \tag{33}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} \gamma_3(x, y)\, dy\, dx \tag{34}$$
$$= -\frac{1}{n^{3/2}}\left(\frac{\alpha_5 - 10\alpha_3}{120}\Phi^{(4)} + \frac{\alpha_3\alpha_4}{72}\Phi^{(6)} + \frac{\alpha_1^3}{216}\Phi^{(8)}\right) + \cdots.$$

Combining (30), (32)–(34) and simplifying, we obtain, as the first four terms of the asymptotic expansion of $F(z)$:

$$\iint_{y-L(x)\leqq z'} (w(x, y) + \gamma(x, y))\, dy\, dx = \Phi - \frac{\alpha_3}{6n^{1/2}}(3z'\Phi^{(1)} + \Phi^{(2)})$$
$$+ \frac{1}{4n}\left\{\frac{\alpha_4 - 3}{6}\Phi^{(3)} + \frac{\alpha_3^2}{9}\Phi^{(9)}\right.$$
$$+ z'\left(\frac{\alpha_3^2}{3}\Phi^{(4)} + \frac{2(\alpha_4 - 1) - \alpha_3^2}{2}\Phi^{(2)} - \frac{\alpha_4 - 1}{2}\Phi^{(0)}\right)$$
$$\left.+ z'^2\left(\frac{\alpha_4 - 1}{2}\Phi^{1)} + \frac{\alpha_3^2}{2}\Phi^{(3)}\right)\right\}$$
$$+ \frac{1}{24n^{3/2}}\left\{-\frac{\alpha_5 - 10\alpha_3}{5}\Phi^{(4)} + \frac{\alpha_3\alpha_4}{3}\Phi^{(6)} + \frac{\alpha_3^3}{9}\Phi^{(8)}\right. \tag{35}$$
$$+ z'\left[\frac{6\alpha_5 - 3\alpha_3 - 9\alpha_3\alpha_4}{2}\Phi^{(1)} + \frac{7\alpha_3(\alpha_4 - 1 - \alpha_3^2) - 2\alpha_5}{2}\Phi^{(3)}\right.$$
$$\left.+ \frac{\alpha_3(\alpha_3^2 - 5\alpha_4 + 7)}{2}\Phi^{(5)} - \frac{\alpha_3^3}{3}\Phi^{(7)}\right]$$
$$+ z'^2\left[\frac{9\alpha_3\alpha_4 + 3\alpha_3 - 6\alpha_5}{2}\Phi^{(2)} + \frac{\alpha_3(3\alpha_3^2 - 7\alpha_4 + 7)}{2}\Phi^{(4)}\right]$$
$$\left.+ z'^3\left[\frac{-3\alpha_3(\alpha_4 - 1)}{2}\Phi^{(3)} - \frac{\alpha_3^2}{2}\Phi^{(5)}\right]\right\} + \cdots.$$

10. In order to estimate the remainder $C(z')n^{-(k-2)/2}$ in the Taylor expansion we write, in accordance with Lemma 3,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} (w(x, y) + \gamma(x, y))\, dy\, dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} \left\{w(x, y) + \sum_{\nu=1}^{k-3}\lambda^{\nu}\Sigma(-1)^{\nu_1+\nu_2}a_{\nu_1\nu_2}w_{\nu_1\nu_2}(x, y)\right\} dy\, dx$$
$$= f_{00}(\lambda) + \sum_{\nu=1}^{k-3}\lambda^{\nu}\Sigma(-1)^{\nu_1+\nu_2}a_{\nu_1\nu_2}f_{\nu_1\nu_2}(\lambda) = \sum_{j=0}^{k-3} f_{00}^{(j)}(0)\frac{\lambda^j}{j!}$$
$$+ f_{00}^{(k-2)}(\theta\lambda)\frac{\lambda^{k-2}}{(k-2)!} + \sum_{\nu=1}^{k-3}\lambda^{\nu}\Sigma(-1)^{\nu_1+\nu_2}a_{\nu_1\nu_2}$$
$$\cdot\left\{\sum_{j=1}^{k-3-\nu} f_{\nu_1\nu_2}^{(j)}(0)\frac{\lambda^j}{j!} + f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)\frac{\lambda^{k-2-\nu}}{(k-2-\nu)!}\right\}$$

$$= B(z') + f_{00}^{(k-2)}(\theta\lambda)\frac{\lambda^{k-2}}{(k-2)!} + \sum_{\nu=1}^{k-3}\Sigma(-1)^{\nu_1+\nu_2}a_{\nu_1\nu_2}f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)\frac{\lambda^{k-2}}{(k-2-\nu)!}$$

$$= B(z') + \Lambda_k\lambda^{k-2}\left(f_{00}^{(k-2)}(\theta\lambda) + \sum_{\nu=1}^{k-3}f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)\right).$$

Thus

$$C(z') = \Lambda_k\left(f_{00}^{(k-2)}(\theta\lambda) + \sum_{\nu=1}^{k-3}f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)\right)$$

Now we may write

$$f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda) = \int_{-\infty}^{\infty}\Sigma(\Pi(g^{(s)}(\theta\lambda))^t)w_{pq}(x, g(\theta\lambda))\,dx$$

where, if we attach a weight s to $g^{(s)}(\theta\lambda)$, the polynomial under the integral sign is isobaric of weight $k - 2 - \nu$ in these $g^{(s)}$'s, and the coefficient of each term is a constant multiple of a certain $w_{pq}(x, g(\theta\lambda))$. Further, it is easily seen by induction that we have

$$g^{(s)}(\theta\lambda) = P_{1+2s}(z)(1 + \theta^2\lambda^2z^2)^{-\frac{1}{2}-s}$$

where $P_{1+2s}(z)$ is a polynomial of the three variables z, x, $\theta\lambda$ which is of at most the $(1 + 2s)$th degree in z and of the $(2l - 1)$st degree in x, and whose coefficients are all Λ_k.

Therefore,

$$|f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)| \leqq \int_{-\infty}^{\infty}Q_k(|x| + x^2 + \cdots + |x|^{2l-1})$$

$$\cdot(1 + |z|^{(1+2s)(k-2-\nu)s^{-1}})w_{pq}(x, g(\theta\lambda))\,dx$$

$$\leqq \int_{-\infty}^{\infty}Q_k(|x| + x^2 + \cdots + |x|^{2l-1})(1 + |z|^{1+2(k-2)})e^{-\frac{1}{2}x^2}\,dx$$

$$\leqq Q_k(1 + |z|^{2k-3})$$

Thus

(36)
$$C(z') \leqq Q_k(1 + |z|^{2k-3})$$

Lastly, an estimate of D is easy:

(37)
$$|F_1'(u)| = \left|\frac{d}{du}\int_{-\infty}^{\infty}\int_{-\infty}^{u+L(x)}(w(x, y) + \gamma(x, y))\,dy\,dx\right|$$

$$\leqq \int_{-\infty}^{x}(w(x, u + L(x)) + |\gamma(x, u + L(x))|)\,dx \leqq Q_k.$$

Collecting the results of (24), (36), (37) we obtain

$$\Pr\{Y - L(X) \leqq z'\} = B(z') + \Lambda_k((1 + |z|^{2k-3})n^{-\frac{1}{2}(k-2)} + (1 + z^2)n^{-\alpha_0}),$$

(38)
$$\alpha_0 = \min\left(\frac{k-2}{2}, \quad \frac{(k-1)\left[\frac{k}{2}\right]}{2\left(\left[\frac{k}{2}\right] + 1\right)}\right).$$

Or, more simply,

$$(39)\qquad | Pr\{Y - L(X) \leqq z'\} - B(z') | \leqq Q_k(1 + |z|^{2k-3})n^{-\alpha_0},$$

where the first four terms of $B(z')$ are given by (35).

12. To return to $F(z)$. We see that $B(z')$ depends on the function $L(x)$. Recalling section 3 we now write B_m for the B corresponding to L_m, with $m = 2l - 1$ or $2l$.

Then by (4) the value of $F(z)$ lies between

$$\Pr\{Y - L_{2l-1}(X) \leqq z'\} \quad \text{and} \quad \Pr\{Y - L_{2l}(X) \leqq z'\}.$$

From the asymptotic expansion just obtained for either of them, we see that the absolute value of their difference does not exceed

$$| B_{2l-1}(z') - B_{2l}(z') | + Q_k(1 + |z|^{2k-3})n^{-\alpha_0}.$$

But

$$L_{2l}(x) = L_{2l-1}(x) - z'b_{2l}(\alpha_4 - 1)^l n^{-l} x^{2l} = L_{2l-1}(x) - b'_{2l}x^{2l} \text{ say,}$$

hence

$$| B_{2l-1}(z') - B_{2l}(z') | \leqq \int_{-\infty}^{\infty} \int_{z'+L_{2l-1}(x)-b'_{2l}x^{2l}}^{z'+L_{2l-1}(x)} |w(x, y) + \gamma(x, y)| \, dy \, dx$$

$$\leqq Q_k b'_{2l} \leqq Q_k |z| n^{-l} < Q_k |z| n^{-\alpha_0}.$$

Therefore

$$| Pr\{Y - L_{2l-1}(X) \leqq z'\} - Pr\{Y - L_{2l}(X) \leqq z'\} | \leqq Q_k n^{-\alpha_0}$$

and so we obtain

$$(40)\qquad F(z) = B(z') + \Lambda_k(1 + |z|^{2k-3})n^{-\alpha_0}$$

which is equivalent to (2) in the theorem stated in section 1.

Thus the theorem will be proved if the assertions regarding the form of $f(z)$ in (1) are shown to be true.

For this purpose we denote, as before, the terms of the order $n^{-\nu/2}$ in $\psi(it_1, it_2)$ and $\gamma(x, y)$ by ψ_ν, γ_ν respectively. Since the term in ψ_ν which yields a w_{pq} with the greatest q is U_3^ν, we have for every w_{pq} in γ_ν the condition $q \leqq 3\nu$. We expand $\int_{-\infty}^{\infty} \int_{-\infty}^{g(\lambda)} \gamma_\nu(x, y) dy dx$ to $k - 3 - \nu$ terms, in which $f_{pq}(0)$, $f'_{pq}(0)$, $\cdots$, $f_{pq}^{(k-3-\nu)}(0)$ occur. In the integrand of $f_{pq}^{(k-3-\nu)}(0)$, e.g., the coefficients of each $w_{pq}(x, z)$ are polynomials in z and x of a total degree in z and x not exceeding that of $(g'(0))^{k-3-\nu}$, i.e., $2(k - 3 - \nu)$. Hence the expansion of γ_ν will give rise to terms of the form

$$z^s I_{pq}^t, \qquad q \leqq 3\nu, \qquad s + t = 2(k - 3 - \nu).$$

Such a term will yield a term $z^s \Phi^{(q+t)}$, which in turn yields the terms $\Phi^{(r)}$ with

$$r \leqq s + q + t \leqq 3\nu + 2(k - 3 - \nu) \leqq 3(k - 3),$$

Equality holds only when $\nu = k - 3$ and $q = 3(k - 3)$. But when $\nu = k - 3$, the term in question is

$$f_{0,3(k-3)}(0) = I^0_{0,3k-10} = \Phi^{(3k-10)}$$

Next, we see that ψ_ν contains $U_3, \cdots, U_{\nu+2}$. Since $f^{(k-3-\nu)}(0)$ is a polynomial of the $(k - 3 - \nu)$th degree in x, the expansion of γ_ν will yield $I^0_{pq}, \cdots, I^{k-3-\nu}_{pq}$. But $I^{k-3-\nu}_{pq} = 0$ if $p > k - 3 - \nu$, hence $p \leqq k - 3 - \nu$. Thus in ψ_ν we need only take account of the terms $(it_1)^p(it_2)^q$ with $p \leqq k - 3 - \nu$. Now if $j < k - 3 - \nu$, in U_j only $\alpha_3, \cdots, \alpha_{2(k-3-\nu)}$ occur. If $j \geqq k - 3 - \nu$, in the coefficient of a term $(it_1)^p(it_2)^q$ with $p \leqq k - 3 - \nu$ the greatest index of α is

$$2(k - 3 - \nu) + j - (k - 3 - \nu) = j + k - 3 - \nu \leqq k - 1$$

since $j \leqq \nu + 2$. Hence in the expansion of every γ only $\alpha_3, \cdots, \alpha_{k-1}$ occur. The proof of the theorem is completed.

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