CERTAIN CELL-LIKE MAPS OF S^4 WITH ZERO-DIMENSIONAL SINGULAR SETS ARE APPROXIMABLE BY HOMEOMORPHISMS. AN EXPOSITION OF A THEOREM OF M. FREEDMAN

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M. Freedman's construction of topological 2-handles in dimension 4 relies heavily on the fact that certain cell-like maps of S^4 are approximable by homeomorphisms. This paper is an exposition of Freedman's proof of this fact.

1. INTRODUCTION TO THE THEOREM

We begin with the necessary definitions.

Let X and Y be compact spaces and let $f : X \to Y$ be a map. The <u>singular set</u> of f, denoted S(f), is the set

$\{y \in Y : f^{-1}(y) \text{ contains more than one point}\}$

Observe that for every $\varepsilon > 0$, the set $\{y \in Y : \operatorname{diam} f^{-1}(y) \geq \varepsilon\}$ is compact. Since $S(f) = \cup_{i=1}^{\infty} \{y \in Y : \operatorname{diam} f^{-1}(y) \geq 1/i\}$, we conclude that S(f) is σ-compact. f <u>can be approximated by homeomorphisms</u> if every neighborhood of f in $X \times Y$ contains a homeomorphism from X to Y (or equivalently, if for every $\varepsilon > 0$, there is a homeomorphism $h: X \to Y$ such that $d(f(x), h(x)) < \varepsilon$ for every $x \in X$).

Let M be an n-manifold. A compact subset S of int M is a <u>tame zero-dimensional</u> subset of M if for every $\varepsilon > 0$, $S \subset \cup_{i=1}^{k} \operatorname{int} B_i$ where

$B_1, B_2, \ldots, B_k$ are disjoint n-cells of diameter $< \varepsilon$ in int M. A σ-compact subset of int M is a tame zero-dimensional subset of M if it is the union of countably many compact tame zero-dimensional subsets.

Let X be a metric space. A subset of X is nowhere dense if its closure has empty interior. Two subsets S and T of X are separated in X if $(\mathrm{cl}\ S) \cap T = \phi = S \cap (\mathrm{cl}\ T)$ (or equivalently, if there are disjoint open subsets U and V of X such that $S \subset U$ and $T \subset V$).

We shall present a proof of the following theorem of M. Freedman.

THEOREM. If $f : S^n \to S^n$ is a surjective map such that S(f) is a nowhere dense tame zero-dimensional subset of S^n, then f can be approximated by homeomorphisms.

We precede the proof with several comments.

This theorem is a slight strengthening of the result that Freedman stated and used in the construction of topological 2-handles in dimension 4. He hypothesized that $\{f^{-1}(y) : y \in S(f)\}$ is a null collection (for every $\varepsilon > 0$, the set $\{y \in S^n : \mathrm{diam}\ f^{-1}(y) \geqslant \varepsilon\}$ is finite) and that S(f) is nowhere dense in S^n. We observed that minor modifications of his argument yield a proof of the above theorem.

In the hypothesis of this theorem, "surjective" can be replaced by "non-constant." For if f is non-constant, then it is a cell-like map, hence a homotopy equivalence, hence surjective.

This theorem is valid for every positive integer n; but it represents new information only when n = 4. In all dimensions other than 4, the theorem is subsumed under the more general cellular approximation theorems of S.

Armentrout (Memoir 107, Amer. Math. Soc. 1971) and L. Siebenmann (Topology 11, 1972, 271-294).

This theorem might be regarded as an extension of M. Brown's Generalized Schoenflies Theorem (Bull. Amer. Math. Soc. 66, 1960, 74-76). Indeed, Brown's method implicitly establishes the following result. If $f : S^n \to S^n$ is a surjective map such that S(f) is a finite set, then f can be approximated by homeomorphisms. Furthermore, the techniques used in the proof of Lemma 4 below are reminiscent of Brown's techniques.

The proof of the theorem is not a traditional decomposition space argument. It is not accomplished by "shrinking" the large point inverses of f in the usual sense. Instead it relies on a replication device which makes the large point inverses of f disappear at the expense of creating other complications. (We comment in more detail immediately before Lemma 4 and at the beginning of the proof of Lemma 6.) The proof has more affinity with M. Brown's Generalized Schoenflies Theorem and L. Siebenmann's Cellular Approximation Theorem (both cited above) than with any decomposition-shrinking argument.

The scheme which employs the above-mentioned replication device imposes a rather interesting complication on the proof. It apparently forces the use of relations which are neither maps nor their inverses. In fact, the approximating hoemomorphism which is the goal of the proof arises as the limit of such relations. For this reason, simple techniques for manipulating relations appear in the proof.

2. FIVE LEMMAS

We now prove five lemmas.

The first three lemmas develop information about tame zero-dimensional sets. We find it useful to begin by recalling some facts about the topology of the homeomorphism space of a compactum.

Suppose X is a compact space with metric ρ. Let $\mathcal{H}(X)$ denote the space of homeomorphisms of X with the compact-open topology. (One basis for the compact-open topology on $\mathcal{H}(X)$ consists of all sets of the form $\{h \in \mathcal{H}(X) : h \subset O\}$ where O varies over the open subsets of $X \times X$.) The compact-open topology on $\mathcal{H}(X)$ is induced by the "supremum metric" σ which is defined by $\sigma(g,h) = \sup\{\rho(g(x),h(x)) : x \in X\}$. Although σ is generally not a complete metric on $\mathcal{H}(X)$, a complete metric τ on $\mathcal{H}(X)$ is easily produced in terms of σ by the formula $\tau(g,h) = \sigma(g,h) + \sigma(g^{-1},h^{-1})$. For a subset A of X, define $\mathcal{H}(X,A) = \{h \in \mathcal{H}(X) : h|A = 1|A\}$. If $A \subset X$, then $\mathcal{H}(X,A)$ is a closed subset of $\mathcal{H}(X)$; hence, the complete metric τ on $\mathcal{H}(X)$ restricts to a complete metric on $\mathcal{H}(X,A)$.

LEMMA 1. If S is a compact tame zero-dimensional subset of the interior of a compact manifold M, and T is a closed nowhere dense subset of M, then $1|M$ can be approximated by homeomorphisms h of M such that $h(S) \cap T = \phi$ and $h|\partial M = 1|\partial M$.

PROOF. Let $\varepsilon > 0$. Enclose S in a finite number of disjoint n-cells of diameter $< \varepsilon$ in int M. Let h squeeze each of these n-cells toward a point in its interior which does not lie in T. ∎

LEMMA 2. Let S be a σ-compact subset of the interior of a compact PL n-manifold M. The following three statements are equivalent.

(1) S is a tame zero-dimensional subset of M.

(2) For every closed nowhere dense subset T of M, $1|M$ can be approximated by homeomorphisms h of M such that $S \cap h(T) = \phi$ and $h|\partial M = 1|\partial M$.

(3) Every compact subset of S is a tame zero-dimensional subset of M.

PROOF. To prove (1) implies (2), let $S = \cup_{i=1}^{\infty} S_i$ where each S_i is a compact tame zero-dimensional subset of M. For each $i \geqslant 1$, let $U_i = \{h \in \mathcal{H}(M,\partial M) : S_i \cap h(T) = \phi\}$. Clearly each U_i is an open subset of $\mathcal{H}(M,\partial M)$, and Lemma 1 implies that each U_i is a dense subset of $\mathcal{H}(M,\partial M)$. Since $\mathcal{H}(M,\partial M)$ has a complete metric, we conclude via the Baire Category Theorem that $\cap_{i=1}^{\infty} U_i$ is a dense subset of $\mathcal{H}(M,\partial M)$. Statement (2) follows because $1|M$ is approximable by elements of $\cap_{i=1}^{\infty} U_i$.

Now assume Statement (2). Let S_0 be a compact subset of S. Choose $\varepsilon > 0$, and let T be the $(n-1)$-skeleton of a triangulation of M whose simplices are all of diameter $< \varepsilon/3$. Statement (2) provides a homeomorphism h of M within $\varepsilon/3$ of $1|M$ such that $S \cap h(T) = \phi$ and $h|\partial M = 1|\partial M$. There is a regular neighborhood N of T in M such that $S_0 \cap h(N) = \phi$. Let $B = h(M - \text{int } N)$. Then $S_0 \subset \text{int } B$ and each component of B is an n-cell of diameter $< \varepsilon$. This proves S_0 is a tame zero-dimensional subset of M.

Clearly (3) implies (1). ∎

LEMMA 3. If S and T are nowhere dense σ-compact tame zero-dimensional subsets of the interior of a compact manifold M, then $1|M$ can be approximated by homeomorphisms h of M such that $h(S)$ and T are separated in M and $h|\partial M = 1|\partial M$.

PROOF. Let $S = \cup_{i=1}^{\infty} S_i$ and $T = \cup_{i=1}^{\infty} T_i$ where each S_i and each T_i is a compact tame zero-dimensional subset of M. For each $i \geqslant 1$, let $U_i = \{h \in \mathcal{H}(M,\partial M) : h(S_i) \cap \mathrm{cl}\ T = \phi\}$ and $V_i = \{h \in \mathcal{H}(M,\partial M) : h(\mathrm{cl}\ S) \cap T_i = \phi\}$. Clearly each U_i and each V_i are open subsets of $\mathcal{H}(M,\partial M)$. Also Lemma 1 implies that each U_i and each V_i are dense subsets of $\mathcal{H}(M,\partial M)$. Since $\mathcal{H}(M,\partial M)$ has a complete metric, we conclude via the Baire Category Theorem that $\cap_{i=1}^{\infty} (U_i \cap V_i)$ is a dense subset of $\mathcal{H}(M,\partial M)$. Lemma 3 now follows because $1|M$ can be approximated by elements of $\cap_{i=1}^{\infty} (U_i \cap V_i)$. ■

The fourth lemma crystallizes the replication device which is the heart of the proof of the theorem. In this lemma, the preimage pattern of the given map ϕ on $\phi^{-1}(A)$ is replicated by a new map ψ on $\psi^{-1}(A)$; and the replication is witnessed by a homeomorphism $\lambda : \phi^{-1}(A) \to \psi^{-1}(A)$ such that $\psi \circ \lambda = \phi|\phi^{-1}(A)$. We foreshadow the proof of the theorem by remarking that this replication allows us to replace the map ϕ by a relation R which is defined by setting $R = \lambda$ on $\phi^{-1}(\mathrm{int}\ A)$, $R = \psi^{-1} \circ \phi$ on $\phi^{-1}(B - \mathrm{int}\ A)$ and $R = \phi$ on $\phi^{-1}(C - \mathrm{int}\ B)$. R represents an improvement over ϕ in that it has no non-trivial point inverses in $\phi^{-1}(A)$. The apparent disadvantage of this procedure is that it exchanges a map for a relation.

We define two n-cells $A \subset B$ to be <u>concentric</u> if $A \subset \mathrm{int}\ B$ and $B - \mathrm{int}\ A$ is homeomorphic to $S^{n-1} \times [0,1]$.

LEMMA 4. Suppose $\phi : C' \to C$ is a map between n-cells which carries $\partial C'$ homeomorphically onto ∂C. Let $A \subset B \subset C$ be concentric n-cells. Then there is a map $\psi : B \to B$ and a homeomorphism $\lambda : \phi^{-1}A \to \psi^{-1}A$ with the following properties.

(1) $\psi|\partial B = 1|\partial B$

(2) $\psi\circ\lambda = \phi|\phi^{-1}A$

(3) If $S(\phi)$ is a nowhere dense tame zero-dimensional subset of C, then so is $S(\psi)$; and if $S(\phi) \cap \partial A = \phi$, then $S(\psi) \cap \partial A = \phi$

(4) If $S(\phi)$ is a nowhere dense tame zero-dimensional subset of C and $\mathrm{cl}\, S(\phi) \subset \mathrm{int}\, C$, then $S(\phi) - A$ and $S(\psi) - A$ are separated in C.

PROOF. The homeomorphism $\phi|\partial C' : \partial C' \to \partial C$ extends by coning to a homeomorphism $\alpha : C' \to C$. A homeomorphism $\sigma : B \to C$ such that $\sigma|A = 1|A$ is easily obtained by sliding in the product structures on B - int A and C - int B. A map $\psi : B \to B$ is defined by $\psi = \sigma^{-1}\circ\phi\circ\alpha^{-1}\circ\sigma$. Since $\alpha|\partial C' = \phi|\partial C'$, then $\psi|\partial B = 1|\partial B$. Since $\sigma|A = 1|A$, then $\psi^{-1}(A) = \sigma^{-1}\circ\alpha\circ\phi^{-1}(A)$. Hence a homeomorphism $\lambda : \phi^{-1}(A) \to \psi^{-1}(A)$ is defined by setting $\lambda = \sigma^{-1}\circ\alpha|\phi^{-1}(A)$. Then $\psi\circ\lambda = \sigma^{-1}\circ\phi|\phi^{-1}(A) = \phi|\phi^{-1}(A)$ because $\sigma^{-1}|A = 1|A$.

Evidently $S(\psi) = \sigma^{-1}(S(\phi))$. Property (3) follows from this.

Now assume $S(\phi)$ is a nowhere dense tame zero-dimensional subset of C and $\mathrm{cl}\, S(\phi) \subset \mathrm{int}\, C$. We modify the proof only in taking more care in the construction of σ. We begin with any homeomorphism $\tau : B \to C$ such that $\tau|A = 1|A$. We apply Lemma 3 in B - int A substituting $\tau^{-1}(S(\phi)) - A$ for S and $S(\phi) \cap (\mathrm{int}\, B - A)$ for T. This yields a homeomorphism h of B which restricts to the identify on $A \cup \partial B$ such that $h\circ\tau^{-1}(S(\phi)) - A$ and $S(\phi) \cap (\mathrm{int}\, B - A)$ are separated in B. Since $\mathrm{cl}\, S(\phi) \subset \mathrm{int}\, C$, then $\mathrm{cl}\, h\circ\tau^{-1}(S(\phi)) \subset \mathrm{int}\, B$. It follows that $h\circ\tau^{-1}(S(\phi)) - A$ and $S(\phi) - A$ are separated in C. Finally define the homeomorphism $\sigma : B \to C$ by $\sigma = \tau\circ h^{-1}$. Since both τ and h restrict to the identity on A, so does σ. Now we have $\sigma^{-1}(S(\phi)) - A$ and $S(\phi) - A$ are separated in C. Since $S(\psi) = \sigma^{-1}(S(\phi))$, property (5) is proved. ■

The fifth lemma concerns relations. It is used in the proof of the theorem to guarantee that the sequence of relations produced there converges to a homeomorphism. Before starting this lemma, we establish some convenient notation for relations which generalizes the usual functional notation.

Let $R \subset X \times Y$; i.e., R is a relation from the set X to the set Y. Define

$$R^{-1} = \{(y,x) \in Y \times X : (x,y) \in R\}$$

If $S \subset Y \times Z$, define

$$S \circ R = \{(x,z) \in X \times Z : (x,y) \in R \text{ and } (y,z) \in S \text{ for some } y \in Y\}$$

If $x \in X$, define $R(x) = \{y \in Y : (x,y) \in R\}$. Thus for $y \in Y$, $R^{-1}(y) = \{x \in X : (x,y) \in R\}$. If $A \subset X$, define $R(A) = \cup \{R(x) : x \in A\}$ and define $R|A = R \cap (A \times Y)$.

LEMMA 5. Let R be a closed subset of $X \times Y$ where X and Y are compact metric spaces. Suppose that $\varepsilon > 0$ and $\operatorname{diam} R(x) < \varepsilon$ for each $x \in X$. Then R has a closed neighborhood N in $X \times Y$ such that $\operatorname{diam} N(x) < \varepsilon$ for each $x \in X$.

PROOF. There is a decreasing sequence $N_1 \supset N_2 \supset N_3 \supset \ldots$ of closed neighborhoods of R in $X \times Y$ such that $\cap_{i=1}^{\infty} N_i = R$. We assert that some N_i has the desired property: $\operatorname{diam} N_i(x) < \varepsilon$ for each $x \in X$. For otherwise, there are sequences $\{(x_i,y_i)\}$ and $\{(x_i,z_i)\}$ in $X \times Y$ such that for each $i \geqslant 1$, (x_i,y_i) and (x_i,z_i) lie in N_i and $\operatorname{diam} \{y_i,z_i\} \geqslant \varepsilon$. Since X and Y are compact, then by passing to subsequences, we can assume that the sequence $\{x_i\}$ converges to the point x in X, and that the sequences $\{y_i\}$ and $\{z_i\}$ converge to the points y and z, respectively, in Y. Consequently, diam $\{y,z\} \geqslant \varepsilon$. Also since $R = \cap_{i=1}^{\infty} N_i$, it follows that (x,y) and (x,z) belong to R. Hence y and z belong to R(x). Since diam $R(x) < \varepsilon$, we have a contradiction. ■

3. THE PROOF OF THE THEOREM

The proof of the theorem is inductive. The induction step, which has a rather complicated statement, is isolated in Lemma 6 below.

We begin by describing the strategy of the proof. Let N_0 be a closed neighborhood of f in $S^n \times S^n$. The goal is to produce a homeomorphism $h : S^n \times S^n$ such that $h \subset N_0$. This will be accomplished by constructing a decreasing sequence $N_0 \supset N_1 \supset N_2 \supset \cdots$ of closed subsets of $S^n \times S^n$ with the property that for each $i \geqslant 1$ and each $x \in S^n$, $N_i(x)$ and $N_i^{-1}(x)$ are non-empty subsets of S^n of diameter $< 1/i-1$. Upon setting $h = \cap_{i=1}^{\infty} N_i$, we see that $h : S^n \times S^n$ is a bijection which is, in fact, a homeomorphism because h is a closed subset of $S^n \times S^n$.

We now give more details. We shall construct the following sequences.

(1) $f_i : S^n \to S^n$ and $g_i : S^n \to S^n$ are surjective maps such that $S(f_i)$ and $S(g_i)$ are nowhere dense tame zero-dimensional subsets of S^n.

(2) R_i is a closed subset of $S^n \times S^n$ such that for each $x \in S^n$, diam $R_i^{-1}(x) < 1/i$ when i is odd, while diam $R_i(x) < 1/i$ when is is even.

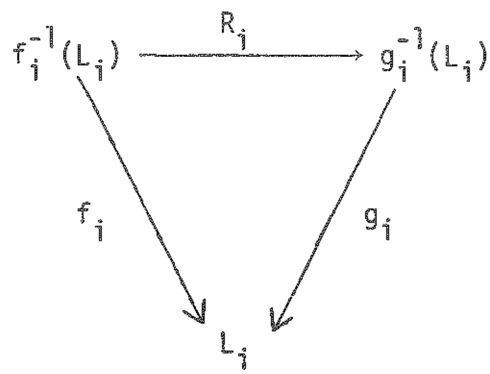
(3) L_i is a compact n-dimensional submanifold of S^n with the following properties.

(a) $R_i \mid f_i^{-1}(S^n - \text{int } L_i)$ is a homeomorphism from $f_i^{-1}(S^n - \text{int } L_i)$ to $g_i^{-1}(S^n - \text{int } L_i)$.

(b) $R_i \mid f_i^{-1}(L_i) = g_i^{-1} \circ f_i \mid f_i^{-1}(L_i)$

(c) $S(f_i) \cap L_i$ and $S(g_i) \cap L_i$ are separated in S^n.

(4) N_i is a closed neighborhood of R_i in $S^n \times S^n$, $N_i \subset N_{i-1}$, and for each $x \in S^n$, diam $N_i^{-1}(x) < 1/i$ when i is odd, while diam $N_i(x) < 1/i$ when i is even.



The construction of these sequences begins with $f_0 = f$, $g_0 = 1|S^n$, $R_0 = f$, $L_0 = S^n$ and N_0 as chosen above.

The construction proceeds inductively.

Given f_{i-1}, g_{i-1}, R_{i-1}, L_{i-1} and N_{i-1}, we obtain f_i, g_i, R_i and L_i via Lemma 6 below. When i is <u>odd</u>: $f_i = f_{i-1}$ and Lemma 6 is applied with $(f_{i-1}, g_{i-1}, R_{i-1}, L_{i-1}, \text{int } N_{i-1}, 1/i)$ substituted for $(f, g, R, L, N, \varepsilon)$; then Lemma 6 produces (g_*, R_*, L_*) which we relabel (g_i, R_i, L_i). Therefore $R_i \subset \text{int } N_{i-1}$. When i is <u>even</u>: $g_i = g_{i-1}$ and Lemma 6 is applied with $(g_{i-1}, f_{i-1}, R_{i-1}^{-1}, L_{i-1}, \text{int } N_{i-1}^{-1}, 1/i)$ substituted for $(f, g, R, L, N, \varepsilon)$; then Lemma 6 produces (g_*, R_*, L_*) which we relabel (f_i, R_i^{-1}, L_i). Again $R_i \subset \text{int } N_{i-1}$.

Next Lemma 5 is used to obtain N_i. When i is <u>odd</u>: Lemma 5 is applied with $(S^n, S^n, R_i^{-1}, 1/i)$ substituted for (X, Y, R, ε); then Lemma 5 provides

N, and we set $N_i = N^{-1} \cap N_{i-1}$. When i is even: Lemma 5 is applied with $(S^n, S^n, R_i, 1/i)$ substituted for (X, Y, R, ε); then Lemma 5 provides N, and we set $N_i = N \cap N_{i-1}$.

Conditions (3a) and (3b) above imply that $R_i(x)$ and $R_i^{-1}(x)$ are non-empty for every $x \in S^n$. Since $R_i \subset N_i$, it follows that $N_i(x)$ and $N_i^{-1}(x)$ are non-empty for every $x \in S^n$. It should now be clear that $N_o \supset N_1 \supset N_2 \supset \ldots$ is a decreasing sequence of closed subsets of $S^n \times S^n$ with the property that for each $i \geqslant 1$ and each $x \in S^n$, $N_i(x)$ and $N_i^{-1}(x)$ are non-empty subsets of S_n of diameter $< 1/i-1$. ∎

LEMMA 6. Suppose $f : S^n \to S^n$ and $g : S^n \to S^n$ are surjective maps such that $S(f)$ and $S(g)$ are nowhere dense tame zero-dimensional subsets of S^n. Suppose R is a closed subset of $S^n \times S^n$ and L is a compact n-dimensional submanifold of S^n with the following properties.

(1) $R \mid f^{-1}(S^n - \text{int } L)$ is a homeomorphism from $f^{-1}(S^n - \text{int } L)$ to $g^{-1}(S^n - \text{int } L)$.

(2) $R \mid f^{-1}(L) = g^{-1} \circ f \mid f^{-1}(L)$.

(3) $S(f) \cap L$ and $S(g) \cap L$ are separated in S^n.

Then for every $\varepsilon > 0$ and every neighborhood N of R in $S^n \times S^n$, there is a surjective map $g_* : S^n \to S^n$ such that $S(g_*)$ is a nowhere dense tame zero-dimensional subset of S^n, there is a closed subset R_* of of $S^n \times S^n$, and there is a compact n-dimensional submanifold L_* of S^n with the following properties.

(1_*) $R_* \mid f^{-1}(S^n - \text{int } L_*)$ is a homeomorphism from $f^{-1}(S^n - \text{int } L_*)$ to $g_*^{-1}(S^n - \text{int } L_*)$.

(2_*) $R_* \mid f^{-1}(L_*) = g_*^{-1} \circ f \mid f^{-1}(L_*)$.

(3_*) $S(f) \cap L_*$ and $S(g_*) \cap L_*$ are separated in S^n.

(4_*) $\text{diam } R_*^{-1}(x) < \varepsilon$ for every $x \in S^n$.

(5_*) $R_* \subset N$.

PROOF. Let $Z = L \cap \{z \in S^n : \text{diam } f^{-1}(z) \geq \varepsilon\}$. Then Z is a compact tame zero-dimensional subset of L. The hypotheses of Lemma 6 imply that $S(f) \cap \partial L = \phi$; so $Z \subset \text{int } L$. It also follows that $\{R^{-1}(y) : y \in S^n \text{ and diam } R^{-1}(y) \geq \varepsilon\} = \{f^{-1}(z) : z \in Z\}$. Thus the elements of $\{f^{-1}(z) : z \in Z\}$ are the sets which must be replaced.

Here is a rough idea of how we proceed. As we remarked in Section 1, we shall not "shrink" the elements of $\{f^{-1}(z) : z \in Z\}$ in the usual sense. Instead, we will enclose Z in a finite number $A_1, A_2, \ldots, A_k$ of small disjoint n-cells; and we shall modify the map g so that for each i, $1 \leq i \leq k$, the preimage pattern of f on $f^{-1}(A_i)$ is replicated by g on $g^{-1}(A_i)$. This will allow us to redefine R on $f^{-1}(A_i)$ so that it carries $f^{-1}(A_i)$ homeomorphically onto $g^{-1}(A_i)$. In this way, the large point inverses of R simply vanish at the expense of complicating the structure of the map g.

There is a finite collection $B_1, B_2, \ldots, B_k$ of disjoint collared n-cells in S^n with the following properties.

(1) $Z \subset \cup_{i=1}^{k} \text{int } B_i$.

(2) $\cup_{i=1}^{k} B_i \subset \text{int } L$.

(3) $(\cup_{i=1}^{k} B_i) \cap \text{cl } S(g) = \phi$.

(4) $f^{-1}(B_i) \times g^{-1}(B_i) \subset N$ for $1 \leq i \leq k$.

(1) is possible because Z is a compact tame zero-dimensional set. (2) is possible because $Z \subset \text{int } L$. Since $S(f) \cap L$ and $S(g) \cap L$ are separated in S^n, then $Z \cap \text{cl } S(g) = \phi$; this makes (3) possible. For each $z \in L$, $f^{-1}(z) \times g^{-1}(z) = R \mid f^{-1}(z) \subset \text{int } N$. Hence, there is a $\delta > 0$ such that $f^{-1}(B) \times g^{-1}(B) \subset \text{int } N$ for any set B of diameter $< \delta$ in L. By choosing the B_i's of diameter $< \delta$, (4) is guaranteed.

Next we find a finite collection $A_1, A_2, \ldots, A_k$ of n-cells in S^n so that

(5) $A_i \subset B_i$ are concentric for $1 \leq i \leq k$, and

(6) $Z \subset \cup_{i=1}^{k} \text{int } A_i$.

We use Lemma 2 to adjust $\cup_{i=1}^{k} \partial A_i$ slightly so that we can also assume

(7) $S(f) \cap \partial A_i = \phi$ for $1 \leq i \leq k$.

Since $S(f)$ and $S(g)$ are nowhere dense in S^n, then for each i, $1 \leq i \leq k$, there is a collared n-cell C_i in S^n so that

(8) $B_i \subset C_i$ are concentric, and

(9) $\text{cl } S(f) \cup \text{cl } S(g) \subset \text{int } C_i$.

Each C_i is the complement of the interior of an appropriately chosen n-cell lying in $S^n - (\text{cl } S(f) \cup \text{cl } S(g))$.

Let $1 \leq i \leq k$. Since $\text{cl } S(f) \subset \text{int } C_i$, then $f^{-1}(\partial C_i)$ is a bicollared (n-1)-sphere in S^n. Therefore, according to the Generalized Schoenflies Theorem, $f^{-1}(C_i)$ is an n-cell. We now apply Lemma 4 to the map $f \mid f^{-1}(C_i) : f^{-1}(C_i) \to C_i$. We thereby obtain a map $\psi_i : B_i \to B_i$ and a homeomorphism $\lambda_i : f^{-1}(A_i) \to \psi_i^{-1}(A_i)$ with the following properties

(10) $\psi_i | \partial B_i = 1 | \partial B_i$.

(11) $\psi_i \circ \lambda_i = f \mid f^{-1}(A_i)$.

(12) $S(\psi_i)$ is a nowhere dense tame zero-dimensional subset of S^n, and $S(\psi_i) \cap \partial A_i = \phi$.

(13) $S(f) - \text{int } A_i$ and $S(\psi_i) - \text{int } A_i$ are separated in S^n.

Lemma 4 actually tells us that $S(f) - A_i$ and $S(\psi_i) - A_i$ are separated in C_i. Property (13) follows because $S(f)$ and $S(\psi_i)$ are disjoint from ∂A_i.

We now define the map $g_* : S^n \to S^n$ by specifying that for $1 \leq i \leq k$,

$$g_* \mid g^{-1}(B_i) = \psi_i \circ g \mid g^{-1}(B_i)$$

and that

$$g_* = g \text{ on } g^{-1}(S^n - \cup_{i=1}^{k} \text{int } B_i)$$

Property (10) guarantees that g_* is well-defined. For $1 \leq i \leq k$, since $g \mid g^{-1}(B_i)$ is a homeomorphism, then the preimage pattern of $g_* \mid g_*^{-1}(A_i)$ replicates the preimage pattern of $f \mid f^{-1}(A_i)$.

Observe that $S(g_*) = S(g) \cup (\cup_{i=1}^{k} S(\psi_i))$. Hence $S(g_*)$ is a nowhere dense tame zero-dimensional subset of S^n.

Define $L_* = L - \cup_{i=1}^{k} \text{int } A_i$. Property (13) together with the fact that $S(f) \cap L$ and $S(g) \cap L$ are separated in S^n imply that $S(f) \cap L_*$ and $S(g_*) \cap L_*$ are separated in S^n.

We define $R_* \subset S^n \times S^n$ by specifying that for $1 \leq i \leq k$

$$R_* \mid f^{-1}(A_i) = g^{-1} \circ \lambda_i$$

and

$$R_* \mid f^{-1}(B_i - \text{int } A_i) = g_*^{-1} \circ f \mid f^{-1}(B_i - \text{int } A_i),$$

and that

$$R_* \mid f^{-1}(S^n - \cup_{i=1}^{k} \text{int } B_i) = R \mid f^{-1}(S^n - \cup_{i=1}^{k} \text{int } B_i).$$

We must verify that R_* is well-defined where the domains of definition overlap; namely on $f^{-1}(\partial A_i)$ and on $f^{-1}(\partial B_i)$ for $1 \leqslant i \leqslant k$. Let $1 \leqslant i \leqslant k$. Since $S(\psi_i) \cap \partial A_i = \phi$, then (11) implies that $\lambda_i \mid f^{-1}(\partial A_i) = \psi_i^{-1} \circ f \mid f^{-1}(\partial A_i)$. Also from the definition of g_* we have $g_*^{-1} \mid \partial A_i = g^{-1} \circ \psi_i^{-1} \mid \partial A_i$. Thus on the set $f^{-1}(\partial A_i)$, we have

$$g^{-1} \circ \lambda_i = g^{-1} \circ \psi_i^{-1} \circ f = g_*^{-1} \circ f.$$

Since $g_* = g$ on $g^{-1}(\partial B_i)$ and since $R = g^{-1} \circ f$ on $f^{-1}(\partial B_i)$, we see that on the set $f^{-1}(\partial B_i)$, $g_*^{-1} \circ f = g^{-1} \circ f = R$.

Evidently $R_* \mid f^{-1}(A_i)$ is a homeomorphism from $f^{-1}(A_i)$ to $g^{-1}(\psi_i^{-1}(A_i)) = g_*^{-1}(A_i)$ for $1 \leqslant i \leqslant k$. Also on $f^{-1}(S^n - \text{int } L)$, R_* coincides with R and, thus, carries $f^{-1}(S^n - \text{int } L)$ homeomorphically onto $g_*^{-1}(S^n - \text{int } L)$. It follows that $R_* \mid f^{-1}(S^n - \text{int } L_*)$ is a homeomorphism from $f^{-1}(S^n - \text{int } L_*)$ to $g_*^{-1}(S_n - \text{int } L_*)$.

On the set $f^{-1}(L - \cup_{i=1}^k \text{int } B_i)$, $R_* = R = g^{-1} \circ f = g_*^{-1} \circ f$. Also $R_* = g_*^{-1} \circ f$ on each of the sets $f^{-1}(B_i - \text{int } A_i)$ for $1 \leqslant i \leqslant k$. It follows that $R_* \mid f^{-1}(L_*) = g_*^{-1} \circ f \mid f^{-1}(L_*)$.

Now let $x \in S^n$. We shall argue that $\text{diam } R_*^{-1}(x) < \varepsilon$. First, if $x \in R_*(f^{-1}(A_i))$ for some i, $1 \leqslant i \leqslant k$, then $R_*^{-1}(x)$ is a point because $R_* \mid f^{-1}(A_i)$ is a homeomorphism. Second, if $x \in R_*(f^{-1}(B_i - \text{int } A_i))$ for some i, $1 \leqslant i \leqslant k$, then $R_*^{-1}(x) = f^{-1}(g_*(x))$ has diameter $< \varepsilon$. This is because $g_*(x) \in B_i - \text{int } A_i$; but f^{-1} takes a point of B_i to a set of diameter $\geqslant \varepsilon$ only if that point lies in $Z \cap B_i$, and $Z \cap B_i \subset \text{int } A_i$. Third, if $x \in R_*(f^{-1}(S^n - \cup_{i=1}^k \text{int } B_i))$, then $R_*^{-1}(x) = R^{-1}(x)$ has diameter $< \varepsilon$, because the point inverses of R of diameter $\geqslant \varepsilon$ all lie in $f^{-1}(Z)$, and $f^{-1}(Z) \subset f^{-1}(\cup_{i=1}^k \text{int } B_i)$.

Let $1 \leqslant i \leqslant k$. Then

$$R_* \mid f^{-1}(A_i) \subset g^{-1} \circ \psi_i^{-1} \circ \psi_i \circ \lambda_i \mid f^{-1}(A_i) = g_*^{-1} \circ f \mid f^{-1}(A_i)$$

Hence

$$R_* \mid f(B_i) \subset g_*^{-1} \circ f \mid f^{-1}(B_i) \subset f^{-1}(B_i) \times g_*^{-1}(B_i)$$

Since $g_*^{-1}(B_i) = g^{-1}(B_i)$, we have

$$R_* \mid f^{-1}(B_i) \subset f^{-1}(B_i) \times g^{-1}(B_i) \subset N$$

Also $R_* \mid f^{-1}(S^n - \cup_{i=1}^{k} \operatorname{int} B_i) \subset R \subset N$. We conclude that $R_* \subset N$. ■