CELEBRATIO MATHEMATICA

S.-S. Chern

HUNG-HSI WU

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Shiing-shen Chern, one of the giants in the history of geometry, passed away in Tianjin, China, on December 3, 2004, at the age of ninety-three. He had spent the last five years of his life in Tianjin, but his career was established mainly in the U.S. from 1949 to 1999.

Chern's work revitalized and reshaped differential geometry and transcendental algebraic geometry. In the decades before 1944 when he embarked on the writing of his historic papers on Chern classes and the geometry of fiber bundles, the field of differential geometry had gone through a period of stagnation. His papers marked the re-entry of differential geometry into the mathematical mainstream, and his tenure at Berkeley (1960–1979) helped make the latter a premier center of geometry in the world. In his lifetime, he had the pleasure of seeing Chern classes become part of the basic vocabulary in contemporary mathematics and theoretical physics. There are also Chern–Weil homomorphism, refined Chern classes for Hermitian bundles, Chern–Moser invariants, and Chern–Simons invariants. Chern was also a mathematical statesman. One does not often see great mathematical insight and great political leadership converge on the same person, but Chern was that rare exception. He was the main liaison between the American and Chinese mathematics communities in the years immediately following the re-opening of China in 1978. Moreover, in the four decades from 1946 to 1984, he founded or co-founded three mathematics institutes; the one in the U.S., the Mathematical Sciences Research Institute, is thriving in Berkeley.

He was born on October 26, 1911, in Jiaxing, Zhejiang Province, China, sixteen days after the revolution that overthrew the Manchurian Dynasty and ushered in modern China. Typically for that era in China, his schooling was haphazard. He had *one day* of elementary education, four years of middle-high school, and at age fifteen skipped two grades to enter Nankai University. Under the tutelage of a great teacher in Nankai, he began to learn some substantive mathematics. To the end of his life, he retained a fond memory of his Nankai years, and this fact was to play a role in his eventual decision to return to China in 1999. His next four years (1930–34) were spent as a graduate student at Qing Hua University in Beijing (then *Peiping*), and he published three papers on the subject of projective differential geometry. While he sensed that his future lay in the general area of geometry, he reminisced in his later years that he was at a loss "how to climb this beautiful mountain." In 1932, Wilhelm Blaschke of Hamburg, Germany, came to visit Beijing and gave some lectures on web geometry. Although those lectures were quite elementary, they opened his eyes to the fact that the era of projective differential geometry had come and gone, and something vaguely called "global differential geometry" was beckoning on the horizon. When he won a

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scholarship in 1934 to study abroad, he defied the conventional wisdom of going to the U.S. and chose to attend the University of Hamburg instead. This is the first of three major decisions he made in the ten-year period 1934–1943 that shaped the rest of his life. As we shall see, the decision in each case was by no means easy or obvious, but it only appeared to be so with hindsight. Throughout his life, he never seemed to lose this uncanny ability to make the right decision at the right time.

Soon after his arrival at Hamburg, he solved one of Blaschke's problems in web geometry and was awarded a doctorate in 1936. However, the most important discovery he made during his two years in Hamburg was the work of Élie Cartan. The discovery was due to not only the fact that Blaschke was one of the few at the time who understood and recognized the importance of Cartan's geometric work, but also the happy coincidence that E. Kähler had just published what we now call the Cartan–Kähler theory on exterior differential systems, and was giving a seminar on this theory at Hamburg. When Chern was given a postdoctoral fellowship in 1936 to pursue further study in Europe, he sought Blaschke's advice. The latter presented him with two choices: either stay in Hamburg to learn algebraic number theory from Emil Artin, or go to Paris to learn geometry from Élie Cartan. At the time, Artin was a major star; he was also a phenomenal teacher, as Chern knew very well firsthand. But Chern made his second major decision by choosing Élie Cartan and Paris. His one-year stay in Paris (1936–37) was, in his own words, "unforgettable." He got to know the master's work directly from the master himself, and Cartan's influence on his scientific outlook can be seen on almost every page of his four-volume *Selected Papers* (1978–1989).

Even before he returned to China in 1937, Chern had been appointed professor of mathematics at Qing Hua University, his former graduate school. Unfortunately, the Sino-Japanese War broke out in Northeast China when he was still in Paris, and Qing Hua University was moved to Kunming in southwestern China as part of the Southwest Associated University. It was to be ten more years before he could set his eyes again on the Qing Hua campus in Beijing. During 1937–43, he taught and studied in isolation in Kunming under harsh war conditions. It must be said that sometimes a little isolation is not a bad thing for people engaging in creative work. For Chern, those years broadened and deepened his understanding of Cartan's work. He wrote near the end of his life that as a result of his isolation, he got to read over 70% of Cartan's papers which total 4,750 pages. Another good thing that came out of those years was his marriage to Shih-ning Cheng in 1939, although a few months later, his pregnant bride had to leave him to return to Shanghai for reasons of personal safety. Their son Paul was born the following year but did not get to meet his father until he was six years old.

In all those years, he kept up his research and his papers appeared in international journals, including two in the *Annals of Mathematics* in 1942. Of the latter, the one on integral geometry [1942b] was reviewed in the *Mathematical Reviews* by André Weil who gave it high praise, and the other on isotropic surfaces [1942a] was refereed by none other than Hermann Weyl, who made this fact known to Chern himself when they finally met 1n 1943. Weyl read every line of the manuscript of [1942a], made suggestions for improvement, and recommended it with enthusiasm. But Chern was not satisfied with just being a known quantity to the mathematical elite because he wanted to find his own mathematical voice. When invitation to visit the Institute for Advanced Study (IAS) at Princeton came from O. Veblen and Weyl in 1943, he seized the opportunity and accepted in spite of the hardship of wartime travel. At considerable personal risk, he spent seven days to fly by military aircrafts from Kunming to Miami via India, Africa, and South

America. He reached Princeton in August by train. The visit to IAS was his third major decision of the preceding decade, and perhaps the most important of all.

His sojourn at the IAS from August 1943 to December of 1945 changed the course of differential geometry and transcendental algebraic geometry; it changed his whole life as well. Soon after his arrival at Princeton, he made a discovery that not only solved one of the major problems of the day — to find an *intrinsic* proof of the n-dimensional Gauss–Bonnet theorem — but also enabled him to define Chern classes on principal bundles with structure group $U(n)$, the unitary group. In the crudest terms, the discovery in question is that, on a Riemannian or Hermitian manifold, the curvature form of the metric can be used to generate topological invariants in a canonical way: certain polynomials in the curvature form are closed differential forms, and are therefore cohomology classes of the manifold via de Rham's theorem. The best entry to this circle of ideas is still Chern's first paper on the subject, the remarkable six-page paper in the *Annals of Mathematics* on the intrinsic proof of the Gauss–Bonnet theorem [1944]. We now turn to a brief description of this paper (cf. [Wu 2008]).

Let M be a compact oriented Riemannian manifold of dimension $2n$. Let a local *frame field* $e_1, \dots, e_{2n}$ be chosen (i.e., the e_i are locally defined vector fields and are orthonormal with respect to the metric). Let $\{\omega^i\}$ be the *dual co-frame field* of $\{e_i\}$ (i.e., the ω^i are 1-forms defined in the same neighborhood as the e_i and $\omega^i(e_j) = \delta^i_j$ for $i, j = 1, \dots, 2n$). Furthermore, let Ω^i_j be the *curvature form* with respect to $\{e_i\}$. Note that if $\tilde{e}_1, \dots, \tilde{e}_{2n}$ is another frame field and h is the orthogonal transition matrix between the frame fields,

$$\tilde{e}_i = \sum_k e_k h^k_i, \tag{1}$$

then the curvature form $\tilde{\Omega}^i_j$ with respect to $\{\tilde{e}_i\}$ satisfies

$$\tilde{\Omega}^i_j = \sum_{\ell,k} (h^{-1})^i_\ell \Omega^\ell_k h^k_j$$

But $(h^{-1})^i_\ell = h^\ell_i$, therefore we have

$$\tilde{\Omega}^i_j = \sum_{\ell,k} h^\ell_i \Omega^\ell_k h^k_j \tag{2}$$

Define the following $2n$-form Ω by

$$\Omega = \frac{1}{2^{2n}\pi^n n!} \sum \varepsilon_{i_1 \dots i_{2n}} \Omega^{i_1}_{i_2} \dots \Omega^{i_{2n-1}}_{i_{2n}} \tag{3}$$

where $\varepsilon_{i_1 \dots i_{2n}}$ is $+1$ or -1, depending on whether $i_1 \dots i_{2n}$ is an even or odd permutation of $1, \dots, 2n$, and is otherwise equal to 0. This Ω is, a priori, dependent on the choice of $\{e_i\}$ and is therefore defined only in the neighborhood where $\{e_i\}$ is defined. Equation (2), however, implies that if $\{e_i\}$ is replaced by $\{\tilde{e}_i\}$, so that Ω^i_j is replaced by the curvature form $\tilde{\Omega}^i_j$ corresponding to $\{\tilde{e}_i\}$, then

$$\sum \varepsilon_{i_1 \dots i_{2n}} \Omega^{i_1}_{i_2} \dots \Omega^{i_{2n-1}}_{i_{2n}} = \sum \varepsilon_{i_1 \dots i_{2n}} \tilde{\Omega}^{i_1}_{i_2} \dots \tilde{\Omega}^{i_{2n-1}}_{i_{2n}}.$$

Therefore the form Ω is independent of the the choice of the frame field $\{e_i\}$ and is a globally defined $2n$-form on M. The Gauss–Bonnet theorem, first proved in complete generality by Allendoerfer–Weil

[1943], states that

$$\int_M \Omega = \chi(M) \tag{4}$$

where $\chi(M)$ denotes the Euler characteristic of M. We shall refer to Ω as the *Gauss–Bonnet integrand*.

The problem with the Allendoerfer–Weil proof is that it is conceptually complex: as the phrase "Riemannian polyhedra" in the title of [Allendoerfer and Weil 1943] suggests, it begins by triangulating M into a simplicial complex with small simplices which are (essentially) isometrically imbeddable into Euclidean space, then integrates the Gauss–Bonnet integrand over each simplex (here earlier results on the Gauss–Bonnet theorem by Fenchel and Allendoerfer for submanifolds in Euclidean space are employed), and then add up the results for the individual simplices carefully to make sure that the boundary terms cancel and the Euler characteristic emerges. One does not know at the end of the proof *why* the theorem is true. Weil conveyed his own misgivings about the proof to Chern upon the latter's arrival at Princeton, and suggested to him that there must be a proof that is *intrinsic* in the sense of not having to appeal to imbedding into Euclidean space. Chern's proof in [1944] achieved exactly this goal, and we proceed to sketch its main ideas.

Consider the *frame bundle* $F(M)$ of M, which is the fibre bundle of orthonormal bases,[1]

$$F(M) = \big\{(x, f_1, \dots, f_{2n}) : x \in M, \text{ and } f_1, \dots, f_{2n} \text{ are an orthonormal basis in the tangent space of } M \text{ at } x\big\}. \tag{5}$$

We have the projection map $\pi : F(M) \to M$. Since Ω is a form on M, the pull-back $\pi^*\Omega$ is a form on $F(M)$. The major step of Chern's proof of the Gauss–Bonnet theorem is that there exists a $(2n-1)$-form Π on $F(M)$ so that

$$\pi^*\Omega = d\Pi \tag{6}$$

Thus the Gauss–Bonnet integrand, when pulled back to $F(M)$, becomes *exact*! This was a totally unexpected result, and was one that underscored for the first time the intrinsic importance of fibre bundles *in differential geometry*. Now sometimes a surprising fact can turn out to be rather trivial because it may only depend on a simple trick, but (6) is quite the opposite. Let θ^i_j be the connection form of the Levi-Civita connection on $F(M)$; θ^i_j $(i, j = 1, \dots, 2n)$ is a 1-form with value in the skew-symmetric matrices $\mathfrak{so}(2n)$, the Lie algebra of the special orthogonal group $\mathrm{SO}(2n)$. The curvature form Θ^i_j on $F(M)$, a 2-form also taking value in $\mathfrak{so}(2n)$, is given by

$$\Theta^i_j = d\theta^i_j + \sum_k \theta^i_k \wedge \theta^k_j.$$

This Θ^i_j is related to the preceding Ω^i_j as follows: If $\{e_i\}$ is the local frame field as before and Ω^i_j is the curvature form with respect to $\{e_i\}$, let e be the local cross-section of $\pi : F(M) \to M$ defined by

$$e(x) = (x, e_1(x), \dots e_{2n}(x)).$$

Then $e^*(\Theta^i_j) = \Omega^i_j$ for all i, j.

1: The use of this bundle in Chern's original proof is implicit as he worked mainly with the unit sphere bundle $S(M)$ of M, but the idea of the proof is the same whether $F(M)$ or $S(M)$ is used.

The last fact about Θ^i_j and Ω^i_j has the following consequence. Consider the $2n$-form Θ defined on $F(M)$ by

$$\Theta = \frac{1}{2^{2n}\pi^n n!} \sum \varepsilon_{i_1 \ldots i_{2n}} \Theta^{i_1}_{i_2} \ldots \Theta^{i_{2n-1}}_{i_{2n}} \tag{7}$$

where $\varepsilon_{i_1 \ldots i_{2n}}$ has the same meaning as before. It follows from (3) that $e^*\Theta = \Omega$, so that $(e \circ \pi)^*\Theta = \pi^*\Omega$. A simple reasoning[2] shows that $(e \circ \pi)^*\Theta = \Theta$. Combining these two relations, we get:

$$\pi^*\Omega = \Theta \tag{8}$$

Thus to prove (6), it suffices to prove

$$\Theta = d\Pi \tag{9}$$

for some $(2n-1)$-form Π on $F(M)$.

Chern's proof of (9) requires the introduction, on $F(M)$, of the following $(2n-1)$-forms $\Phi_0, \Phi_1, \ldots, \Phi_{n-1}$, and the $2n$-forms $\Psi_0, \Psi_1, \ldots, \Psi_{n-1}$: for each $k = 0, \ldots n-1$,

$$\Phi_k = \sum \varepsilon_{i_1, \ldots, i_{2n-1}} \Theta^{i_1}_{i_2} \wedge \cdots \wedge \Theta^{i_{2k-1}}_{i_{2k}} \wedge \theta^{i_{2k+1}}_{i_{2n}} \wedge \cdots \wedge \theta^{i_{2n-1}}_{i_{2n}}$$

and

$$\Psi_k = (2k+1) \sum \varepsilon_{i_1, \ldots, i_{2n-1}} \Theta^{i_1}_{i_2} \wedge \cdots \wedge \Theta^{i_{2k-1}}_{i_{2k}} \wedge \Theta^{i_{2k+1}}_{i_{2n}} \wedge \theta^{i_{2k+2}}_{i_{2n}} \wedge \cdots \wedge \theta^{i_{2n-1}}_{i_{2n}}$$

where each sum is over all permutations $i_1, \ldots, i_{2n-1}$ of $1, \ldots, 2n-1$, and $\varepsilon_{i_1, \ldots, i_{2n-1}}$ is equal to $+1$ or -1, depending on whether the permutation is even or odd. Note that from (7), we have

$$\Psi_{n-1} = (2^{2n}\pi^n n!)\Theta \tag{10}$$

Using the Bianchi identity and the definition of Θ^i_j in terms of the connection form θ^i_j, one obtains the following recurrence relation:

$$d\Phi_k = -\Psi_{k-1} + \frac{2n-2k-1}{2(k+1)}\Psi_k \tag{11}$$

where $k = 0, \ldots, n-1$ and $\Psi_{-1} \equiv 0$ by definition. The sought-after $(2n-1)$-form Π on $F(M)$ is now defined to be

$$\Pi = \frac{1}{\pi^n} \sum_{k=0}^{n-1} \frac{1}{1 \cdot 3 \cdot 5 \ldots (2n-2k-1) \cdot 2^{n+k} k!} \Phi_k \tag{12}$$

Using (11), and then (10), we finally get

$$d\Pi = \frac{1}{2^{2n}\pi^n n!}\Psi_{n-1} = \Theta,$$

which is exactly (9).

This proof of (9) is the envy and despair of all who work in differential geometry. Chern did this computation mainly in his head,[3] and all through his life, he seemed to be able to conjure at will the same magical quality in his computations.

2: Using the fact that Θ is a horizontal form on $F(M)$.

3: See Wu [2005] for more background information.

For the concluding step in the Chern proof of (4), we have to bring in the *sphere bundle*

$$S(M) = \big\{(x, f) : f \text{ is a unit vector in the tangent space of } M \text{ at } x\big\}.$$

$F(M)$ is a fibre bundle over $S(M)$ and we have a natural projection $\pi_1 : F(M) \to S(M)$. Briefly, the forms Φ_k and Ψ_k actually *descend* to $S(M)$, in the sense that they are the pull-backs of forms in $S(M)$ by π_1^*. The same is therefore true of Π and Θ, so that we may regard (9) as a relation between forms on $S(M)$. Now given any point x_0 in M, the Hopf theorem on vector fields says there is a unit vector field v defined in $M \setminus \{x_0\}$ so that its isolated singularity at x_0 has index equal to $\chi(M)$, the Euler characteristic of M. Regarding v as a cross-section of the bundle $S(M) \to M$ over $M \setminus \{x_0\}$, it is elementary to see that, on $M \setminus \{x_0\}$,

$$\Omega = d(v^*\Pi)$$

Moreover, and this is a critical observation due to Chern, the restriction of Π (as a form on $S(M)$) to the fibre S_{x_0} of $S(M)$ over x_0 is exactly

$$\Pi\,|_{S_{x_0}} = \frac{(n-1)!}{2\pi^n} d\sigma$$

where $d\sigma$ is the volume form of the unit sphere S_{x_0} in the tangent space of M at x_0.[4] By a standard argument, expressing $M \setminus \{x_0\}$ as the limit of M minus the small ball of radius ε around x_0 as $\varepsilon \to 0$ and using Stokes' theorem, the integral of (4) becomes the integral

$$\int_{S_{x_0}} v^*\Pi = \int_{v_*(S_{x_0})} \Pi = \chi(M)$$

where we have made use of the classical fact that the volume of the unit sphere in $2n$-dimensional space is $2\pi^n/(n-1)!$. *Now we see exactly why the Gauss–Bonnet theorem is true.*

We may interpret the preceding proof in the following way. The form Ω, being a top degree form on M, is automatically closed and therefore represents a cohomology class by de Rham's theorem. The Gauss–Bonnet theorem (4) says is that this class is the Euler class. Here then is the first example of a canonical representation of a cohomology class by the curvature form of a Riemannian metric. Once this is realized, the next step is perfectly obvious, i.e., how to generalize this construction. The fact that this was Chern's thinking can be inferred not only from his paper [1946a] which introduces Chern classes, but more explicitly from what he said concerning the Gauss–Bonnet theorem in the last sentence of the second paragraph on p. 85 of [1946a] and also from pp. 114–115, *loc. cit.* Before commenting further on [1946a], however, let us pause to make a few historical remarks.

The whole idea of using the curvature form on a principal bundle to generate characteristic classes is now so standard that it is difficult for us, sixty years after the fact, to fully appreciate the startling originality of Chern's contribution. The fact revealed by (6), to the effect that in differential geometry, the associated bundles of a manifold are part and parcel of any attempt to understand the manifold itself, was unimagined at the time. The use of the curvature form and de Rham's theorem to generate cohomology classes was equally revelatory. Perhaps the words of a contemporary, André Weil, can more accurately give a sense of

4: The generator of the cohomology ring of the fibre (a sphere) is thus *transgressive*, in the sense that it is represented by the restriction of a form on $S(M)$ whose exterior derivative is the pull-back of a form from the base M. This is the first appearance of transgression in algebraic topology.

Chern's accomplishment. Weil was among the first in his generation to recognize the significance of Élie Cartan's work, and was familiar with Cartan's theory of exterior differential forms as well as Cartan's use of fibre bundles. In fact, Weil originally wanted to write the Allendoerfer–Weil paper [1943] using differential forms instead of tensors (Weil [1979, p. 554]). Thus he had every advantage a mathematician could ask for to decipher the Gauss–Bonnet enigma, but the insight that there would be a vast conceptual simplification of the Gauss–Bonnet integrand by use of the sphere bundle (in the form of (6)), and that the integrand is a representative of a cohomology class eluded him. As he noted:

> Les espaces fibrés ... Leur rôle en géométrie différentielle, et tout particulièrement dans l'oeuvre d'Élie Cartan a été longtemps resté implicite, mais s'était clarifié peu à peu grâce aux travaux d'Ehresmann et surtout à ceux de Chern. La démonstration par Chern de la formule de Gauss–Bonnet et sa découverte des classes caractéristiques des variétés à structure complexe ou quasi-complexe avaient inauguré une nouvelle époque en géométrie différentielle, ... [Weil 1979, p. 566]

> [Chern and I] were both beginning to realize the major role which fibre bundles were playing, still mostly behind the scenes, in all kinds of geometric problems. ... I will merely point out what can now be realized in retrospect about Chern's proof for the Gauss–Bonnet theorem, as compared with the one Allendoerfer and I had given in 1942, following the footsteps of H. Weyl and other writers. The latter proof, resting on the consideration of "tubes," did depend (although this was not apparent at the time) on the construction of a sphere-bundle, but of a non-intrinsic one, viz., the transversal bundle for a given immersion in Euclidean space; Chern's proof operated explicitly for the first time with an intrinsic bundle, the bundle of tangent vectors of length 1, thus clarifying the whole subject once and for all. [Weil 1978, p. x–xi]

These passages may also shed some light on why Weil's admiration of Chern never flagged throughout his life.

It was already mentioned that Chern began his quest for defining general characteristic classes almost as soon as he saw how to prove the Gauss–Bonnet theorem. To cut a long story short, the result of this work is the substance of his paper [1946a]. Briefly, let a Hermitian metric be given on an n-dimensional complex manifold M, and let Ω^i_j $(i, j = 1, \dots n)$ be the curvature form of the Hermitian connection relative to a local unitary frame field $\{e_i\}$ $(i = 1, \dots n)$ (i.e., each e_i is a vector field of type $(1, 0)$, and $\{e_i(x)\}$ is an orthonormal basis of the holomorphic tangent space at x for each x with respect to the Hermitian metric). Ω^i_j is of type $(1, 1)$. Consider now the following n differential forms $c_k(\Omega)$ $(k = 1, \dots n)$ of type (k, k):

$$c_k(\Omega) = \left(\frac{\sqrt{-1}}{2\pi}\right)^k \sum_{i_1<i_2<\cdots<i_k} \sum_{\sigma} \varepsilon(\sigma)\, \Omega^{i_1}_{\sigma(i_1)} \wedge \dots \Omega^{i_k}_{\sigma(i_k)} \tag{13}$$

where each σ ranges through all permutations of $i_1, \dots, i_k$, and the corresponding $\varepsilon(\sigma)$ is the sign of the permutation, which is $+1$ if σ is even, and -1 if σ is odd. These are the *Chern forms* of the hermitian metric. One argues as in (3) above that these $c_k(\Omega)$ do not depend on the choice of the unitary frame field $\{e_i\}$ so that they are globally defined differential forms on M. A computation using the Bianchi identity shows that in fact each $c_k(\Omega)$ is a closed form. By de Rham's theorem, each $c_k(\Omega)$ represents a

cohomology class of degree $2k$, the k-th *Chern class* of the manifold M. The unusual looking coefficient in (13) guarantees that the Chern classes are integral classes.

Now let

$$U(M) = \{(x, u_1, \dots, u_n) : x \in M, \text{ and } u_1, \dots, u_n \text{ are an orthonormal basis of the holomorphic tangent space of } M \text{ at } x\}. \quad (14)$$

We shall refer to $U(M)$ as the *bundle of unitary frames* over M. We have the natural projection $\pi : U(M) \to M$. The analogue of (6) is that each of these forms $c_k(\Omega)$, when pulled back to $U(M)$, becomes an exact form.[5] As in (12), this fact is proved by an explicit construction:

$$\pi^* c_k(\Omega) = d(Tc_k(\Omega)) \quad (15)$$

where each $Tc_k(\Omega)$ is a form explicitly constructed from $c_k(\Omega)$. For simplicity, we shall refer to $Tc_k(\Omega)$ as the *transgression* of $c_k(\Omega)$.[6] Below, we shall have occasion to refer to the fact that each $Tc_k(\Omega)$ can be written down explicitly in terms of $c_k(\Omega)$ and the metric.

When the Hermitian metric is Kählerian, Chern identified the n-th Chern form $c_n(\Omega)$ with the Gauss–Bonnet integrand of the underlying *Riemannian* manifold of M [1946a, pp. 114–5]. Thus one sees the direct link between the papers [1944] and [1946a]. (As is well known, the n-th Chern *class* is always the Euler class; see [Milnor and Stasheff 1974].) Moreover, Chern's definition of the forms $c_k(\Omega)$ in (13) is based on the fact that the polynomials corresponding to the $c_k(\Omega)$ generate the invariant polynomials of the unitary group. Thus in Chern's seminal work, we see the key ingredients of Weil's 1949 definition of the aptly named *Chern–Weil homomorphism* on a general fibre bundle with an arbitrary Lie group as structure group [Weil 1979, pp. 422–436].

To round off the picture, it should be pointed out that the analogue of the Chern forms for the orthogonal group was introduced around the same time by Pontryagin [1944], though the details came later [1949].

The topology of the forties was preoccupied with the real category, and Chern's work on the characteristic classes of complex manifolds appeared at first to be slightly out of step with the times. But the dramatic growth of algebraic geometry, particularly transcendental algebraic geometry, beginning with the fifties made him a prophet. Chern classes are important in algebraic geometry for at least two reasons. One is that the Chern classes of algebraic varieties suggested that they might furnish a firm foundation for the (then) confusing plethora of algebraic-geometric invariants, and Hodge was among the first to push for this point of view [Hodge 1951]. Chern himself made important contributions in this direction, but F. Hirzebruch's work in the fifties capped this development and made this vision a reality [Hirzebruch 1956]. A second and perhaps more important reason is that, many by-now standard arguments in algebraic geometry (e.g., those using the Kodaira vanishing theorem or applications of Yau's solution of the Calabi Conjecture) are simply not possible without the curvature representations of the Chern classes of a bundle.

5: Of course we recognize with hindsight that this is a reflection of the topological triviality of the total space of the universal bundle.

6: This is the terminology of Chern in his last years, and it differs from the standard usage. However, the precise phenomenon of transgression (without the name) appeared for the first time in Theorem 8 on p. 103 of [1946a].

Chern's fame began to spread after 1944, though slowly, in the American mathematics community, and he was invited to give a one-hour address in the 1945 summer meeting of the American Mathematical Society. In reviewing the text of that address [Chern 1946b], Heinz Hopf wrote in *Mathematical Reviews* that Chern's work had ushered in a new era in global differential geometry. Thereafter, the global study of manifolds became the main direction of geometric research. At age thirty-four, he had realized his youthful dream by scaling one of the highest peaks on that "beautiful mountain."

In April of 1946, Chern returned to China and was immediately entrusted with the creation of a mathematics institute for Academia Sinica in Nanking. That he did, and became its de facto director (the official title was "Deputy Director"). We normally envision a "mathematics institute" to be a gathering of scholars to explore the frontiers of research, but China was not yet ready for that kind of institute for lack of a sufficient number of such Chinese mathematical scholars. Being a realist from beginning to end, Chern turned the institute into the only thing it could have been, namely, China's first true graduate school in mathematics. He recruited a group of young people and personally took charge of their education by teaching them the fundamentals of modern mathematics. Many of this group subsequently became leaders of the next generation of Chinese mathematicians.

By late 1948, the political situation in China had become so unstable that Veblen and Weyl began to be concerned about Chern's safety. With the help of R. Oppenheimer, then director of IAS, Chern and his family managed to land safely on U.S. soil on New Year's Day of 1949. He was to be a member of IAS for the spring semester and, in the fall, take up a faculty position at the University of Chicago where he would stay until 1960. In 1950, he gave a one-hour address at the International Congress of Mathematicians (held in Cambridge, Massachusetts) on the differential geometry of fibre bundles. It was in the decade of the fifties that Chern classes began to force their way into most mathematicians' consciousness, due in no small part to the spectacular advances in algebraic geometry made by Kodaira, Hirzebruch, and others.

In 1960, Chern accepted the offer to come to the University of California at Berkeley. Upon his arrival, he immediately attracted a group of young geometers, and Berkeley in the sixties and seventies became the de facto geometry center of the world. Although he officially retired in 1979, he remained active in Berkeley's departmental affairs until the mid-eighties, and made Berkeley his home until 1999. Many honors came his way during the Berkeley years, the principal ones being the election to the National Academy of Sciences in 1961, the U.S. National Medal of Science in 1975, and the Wolf Prize from the Israel government in 1984. Later, he also received the Lobachevsky Prize from the Russian Academy in 2002, and the first Shaw Prize in mathematics in 2004, a few months before his death. In 2002, he was Honorary President of the International Congress of Mathematicians held at Beijing.

Chern's leadership position in differential geometry was, if anything, enhanced by his work in his Berkeley years. Two of his major papers in this period hark back to his early work on characteristic classes. On the latter, he was wont to point out that his main contribution to characteristic classes was not so much the introduction of Chern classes as the discovery of explicit differential *forms* that represent those *classes*. To him, it was the forms that give geometers an edge over topologists in studying many aspects of these classes. With examples like Yau's solution of the Calabi Conjecture in mind, one can hardly disagree with him. The two pieces of work to be discussed further justify his point of view. In his collaboration with Raoul Bott [1965] on generalized Nevanlinna theory in higher dimensions, they constructed for the

holomorphic category the "correct" version of transgression (cf. (15)) in the top dimension by proving that, in case of an n-dimensional holomorphic vector bundle $\pi : E \to M$ over an n-dimensional complex manifold M, the pull-back of the top Chern form $\pi^* c_n(\Omega)$ to $E \setminus 0$ (here 0 stands for the zero section) is not only exact (see (15)), but "doubly exact":

$$\pi^* c_n(\Omega) = dd^c \rho$$

for some $(n-1, n-1)$ form ρ on $E \setminus 0$. The nontriviality of this assertion comes from the fact that $E \setminus 0$ is neither compact nor assumed to be Kählerian. This property of the top Chern class is crucial for their generalization of Nevanlinna's first main theorem. Along the way, they also made use of this "doubly exact" phenomenon to introduce the *refined Chern classes* which have since found their way into algebraic number theory. Incidentally, this Bott–Chern paper is also a natural extension of Chern's ground-breaking work of the fifties to geometrize Nevanlinna theory by transplanting it to complex manifolds [1960]. The geometric point of view towards Nevanlinna theory has proven to be extraordinarily fruitful in algebraic geometry, and it has repercussions in number theory as well.

The second paper related to Chern's earlier work on characteristic classes dates from 1971, when he and Jim Simons introduced the Chern–Simons invariants [1974]. Let M be an n-dimensional Riemannian manifold. We will be appealing to the Chern–Weil homomorphism, so let $P(u^i_j)$ be an invariant polynomial on the Lie algebra $\mathfrak{so}(n)$ of the orthogonal group $O(n)$. If Θ^i_j is the curvature form on the frame bundle $F(M)$ as in equation (7) above, then $P(\Theta^i_j)$ is a closed from on $F(M)$ that is the pull-back of a form on M (compare equation (8)) and therefore represents a cohomology class of M. Moreover, as in equations (6) and (15), $P(\Theta^i_j)$ is actually an exact form on $F(M)$, which we write as

$$P(\Theta^i_j) = dTP(\Theta^i_j) \tag{16}$$

As mentioned earlier, the form $TP(\Theta^i_j)$ is obtained from $P(\Theta^i_j)$ by an explicit construction. *If the form* $P(\Theta^i_j)$ *is equal to 0*, then $TP(\Theta^i_j)$ becomes a closed form on $F(M)$, and therefore defines a cohomology class of $F(M)$ (rather than M itself).

So far, the forms $P(\Theta^i_j)$ and $TP(\Theta^i_j)$ depend on the choice of the Riemannian metric on M. Now suppose we change the metric on M by a conformal factor, then there is a natural bundle isomorphism between the two frame bundles. In particular, the cohomology groups of their total spaces are naturally isomorphic and will henceforth be identified. With this understood, Chern and Simons proved that, under such a conformal change of metric, the *form* $P(\Theta^i_j)$ *does not change*, and the *cohomology class* of $TP(\Theta^i_j)$, as a class on $F(M)$, does not change either. This cohomology class $[TP(\Theta^i_j)]$ is then a conformal invariant of the Riemannian metric. They went on to give applications of this fact to conformal immersions into Euclidean space.

In case M is a 3-dimensional Riemannian manifold and P_1 is the first Pontryagin polynomial, then $P_1(\Theta^i_j)$, being the pull-back of a 4-form on a 3-dimensional manifold, must be 0. The above considerations therefore apply, and we have a cohomology class $[TP_1(\Theta^i_j)]$ on $F(M)$ which is a conformal invariant of M. But in this case, the *form* $TP_1(\Theta^i_j)$ can be simply written down: with θ^i_j as the connection form on

$F(M)$ (see the notational setup preceding equation (7)),

$$TP_1(\Theta^i_j) = \frac{1}{4\pi^2}\{\theta^1_2 \wedge \theta^1_3 \wedge \theta^2_3 + \theta^1_2 \wedge \Theta^1_2 + \theta^1_3 \wedge \Theta^1_3 + \theta^2_3 \wedge \Theta^2_3\} \tag{17}$$

Much to the surprise of Chern and Simons, physicists in superconductivity and super-string theory both embraced almost immediately the *Chern–Simons action* defined by the 3-form $TP_1(\Theta^i_j)$. Taken by itself, it is a closed 3-form which can be defined for any connection on M without any reference to a metric, and it has continued to play an important role in theoretical physics. This is a dramatic confirmation of Chern's belief in the importance of the forms themselves.

The Chern–Simons invariants cannot be defined unless we have a Pontryagin *form* equal to 0. This naturally raises the question of whether on a given manifold with a vanishing Pontryagin class, there is a Riemannian metric whose corresponding Pontryagin form is zero.

One more major piece of work that Chern did in his Berkeley years should not go unmentioned. In 1974, he and Jürgen Moser wrote a paper in a completely different direction [1974]. Generalizing Élie Cartan's work on real hypersurfaces of complex Euclidean space of dimension two, they defined what we now call the Chern–Moser invariants of such hypersurfaces in all dimensions. These invariants are a complete set of local invariants in the real analytic case. The study of these invariants is now a fundamental part of geometric complex analysis. Finally, in 1992, when he was already eighty, he found inspiration in his own work in the late forties and, with D. Bao and Z. Shen, made a strong advocacy for generalizing classical Riemannian geometry to the Finsler setting. This advocacy has attracted a following.

During his Berkeley years, his leadership was felt in other areas too, but none more so than in the founding of two mathematics institutes. In 1981, the proposal he made jointly with Calvin Moore and I. M. Singer to establish an institute in mathematics on the Berkeley campus was officially approved by the government, and the Mathematics Sciences Research Institute (MSRI) was born. Chern served as its first director until 1984. The operational model of MSRI differs significantly from the most eminent research institute of our time, the Princeton Institute for Advanced Study. In contrast with the latter, MSRI has no permanent faculty, and each year its activities are organized around clearly defined mathematical topics. Senior mathematicians in each topic area are invited to visit MSRI for (part of) the year to help organize the scientific activities. This model has been followed around the world by other institutes since then.

Starting in the seventies, Chern took the lead in re-establishing mathematical communications between the U.S. and China. After his official retirement from the University in 1979, his visits to China became more frequent. Given that China has venerated scholarship for three thousand years, it was easy for someone with Chern's diplomatic skills and preeminence to function smoothly at the highest political level in China. This may partially explain how he was able to establish, in 1984, a mathematics research institute in his alma mater, Nankai University, in Tianjin. A main goal of the Nankai Institute has been to attract leading mathematicians around the world to visit Tianjin and make it an active center of mathematics. Chern pursued this goal with vigor, and the Chinese government did its share in making foreign visitors welcome. When Chern finally returned to China for good in 1999, the well-being of the institute became his final project. He made ambitious plans that were only partially realized at the time of his death.

Chern is survived by his son Paul L. Chern, daughter May P. Chu, and four grandchildren, Melissa, Theresa, Claire, and Albert. His wife of sixty years, Shih-Ning, passed away earlier in year 2000 in Tianjin.

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