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Michael H. Freedman

Slava Krushkal

THE A-B SLICE PROBLEM

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SLAVA KRUSHKAL

The A-B slice problem is a formulation of the 4-dimensional topological surgery conjecture, the main open problem remaining in the geometric classification theory of topological 4-manifolds since Mike Freedman's famous proof in the simply connected case [1982]. He also showed that surgery works for fundamental groups of polynomial growth [Freedman 1984]. Since the early 1980s, the class of groups for which surgery is known to hold ("good groups") has been extended somewhat to groups of subexponential growth [Freedman and Teichner 1995; Krushkal and Quinn 2000]. The class of good groups is closed under the operations of taking subgroups, extensions and direct limits, and all currently known good groups are amenable.

The conjecture of Mike [1984], dating back to 1983, asserts that surgery fails for non-Abelian free groups. It is not difficult to see that the validity of surgery for free groups would in fact imply validity for all groups. Indeed, the question is whether a given surgery kernel — a hyperbolic pair $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$ in $\pi_2(M^4)$ — may be represented by embedded spheres intersecting in a single point. The algebraic intersection numbers are counted in $\mathbb{Z}[\pi_1]$, so geometrically the starting data is a pair of immersed 2-spheres in a 4-manifold M, whose self-intersections and extra intersections are paired up so that the Whitney loops are trivial in $\pi_1(M)$. A regular neighborhood N of the 2-complex given by the 2-spheres together with the Whitney disks is itself a surgery problem with free fundamental group; clearly, solving all "canonical" problems N constructed in this way would imply a solution of an arbitrary surgery problem.

Another collection of canonical surgery problems is constructed as follows: Consider the Borromean rings (and more generally the family of links given by iterated Bing doublings of the Hopf link, where one can also take parallel copies of the components at each stage of the iteration). Then, the surgery conjecture is equivalent to the question of whether the untwisted Whitehead doubles of the links in this collection (Figure 1) are freely slice. Here, a link is *freely slice* if it is topologically slice, and moreover the fundamental group of the slice complement in $\mathbb{D}^4$ is freely generated by meridians to the slices. Considering the slice complement, one observes that this question (say, for the simplest link in this collection, pictured in Figure 1) is equivalent to the existence of a topological 4-manifold M homotopy equivalent to the wedge of three circles and whose boundary is given by the zero-framed surgery on the Whitehead double of the Borromean rings, Wh(*Bor*). Mike's conjecture [1984] is that such M does not exist; proving this would exhibit a counterexample of the surgery conjecture.

In [Freedman 1986b;1986a], Mike introduced an approach to this conjecture, known as the *A-B slice problem*. This approach replaces the slice problem for a very "weak" link Wh(*Bor*) by a generalized slice problem for a much more "robust" link, the Borromean rings. More concretely, suppose the hypothetical
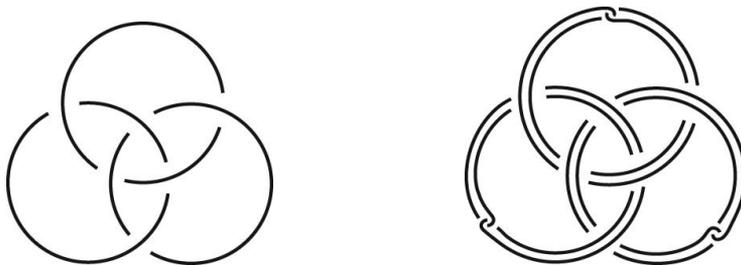
Figure 1. The Borromean rings and their untwisted Whitehead double.

4-manifold M, discussed above, exists. Then, consider its universal cover $\widetilde{M}$. It is shown in [Freedman 1986b] that the end-point compactification of $\widetilde{M}$ is homeomorphic to the 4-ball. The group of covering transformations (the free group on three generators) acts on $\mathbb{D}^4$ with a prescribed action on the boundary, and roughly speaking the A-B slice problem is a program for finding an obstruction to the existence of such actions. Figure 2 illustrates a fundamental domain for this action on $\mathbb{D}^4$ where, for each $i = 1, 2, 3$, the free generator g_i of the covering translation group takes A_i to the complement $\mathbb{D}^4 \smallsetminus B_i$.

To describe the problem in more detail, consider *decompositions* $\mathbb{D}^4 = A \cup B$ of the 4-ball into two codimension-zero smooth submanifolds A and B, extending the standard genus-one Heegaard decomposition of the 3-sphere, $\partial \mathbb{D}^4 = \mathbb{S}^1 \times \mathbb{D}^2 \cup \mathbb{D}^2 \times \mathbb{S}^1$. The components of the Hopf link $\mathbb{S}^1 \times 0 \sqcup 0 \times \mathbb{S}^1$ are thought of as the "attaching curves" of A and B. The submanifolds A and B are disjoint except for their boundary.

An n-component link L is *A-B slice* if there exist n decompositions of the 4-ball, $\mathbb{D}^4 = A_i \cup B_i$ for $i = 1, \ldots, n$, and disjoint embeddings of all $2n$ pieces $\{A_i, B_i\}$ into $\mathbb{D}^4$ so that the attaching curves of the $\{A_i\}$ form the link L, and the attaching curves of the $\{B_i\}$ form a parallel copy of L. Moreover, the covering group action is encoded in the further requirement that each one of the embeddings of A_i, B_i is isotopic to the original embedding, which is specified by the decomposition $\mathbb{D}^4 = A_i \cup B_i$.

Therefore, the surgery conjecture is reformulated into the question of whether the Borromean rings (and a family of their generalizations, discussed above) are A-B slice. An easy argument, using Alexander duality, implies that the Hopf link is not A-B slice, and one is led to believe that a variant of such an argument using Milnor's $\bar{\mu}$-invariants [Milnor 1954] would give an obstruction for the Borromean rings as well. However, the step from the linking number to $\bar{\mu}_{123}$ turns out to be challenging. A particular issue that comes up is the indeterminacy of the higher Milnor's invariants; while usually it is not hard to analyze specific given examples of decompositions, it has been hard to formulate an invariant working uniformly for all decompositions.

The A-B slice approach to surgery was further developed in Mike's joint paper with Xiao-Song Lin [1989]. This paper introduced a collection of *model decompositions* that appeared to approximate, in a certain algebraic sense, an arbitrary decomposition $\mathbb{D}^4 = A \cup B$. It seemed reasonable to think then that, if a suitable obstruction is formulated for these models, one should be able to extend it to all decompositions. [Freedman and Lin 1989] also developed the *relative-slice* reformulation of the problem, useful in particular for analyzing specific examples of decompositions.
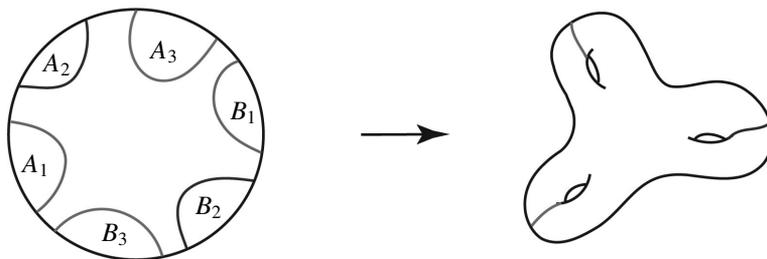
Figure 2. The universal cover of M.

Using a generalization of the Milnor group, an obstruction was given for model decompositions in [Krushkal 2005]. A subtle phenomenon was observed in [Krushkal 2008] which showed that there exist decompositions $\mathbb{D}^4 = A_i \cup B_i$ and disjoint embeddings of the six pieces in the 4-ball with the required boundary conditions, therefore establishing that the Borromean rings are *weakly A-B slice*. This disproved a stronger version of the conjecture of Mike, stated in [Freedman and Lin 1989]; however, these examples do not solve the A-B slice problem in the affirmative, because the equivariance condition is not satisfied: the re-embeddings of A_i, B_i are not standard (that is, are not isotopic to the original ones).

More recently, the notion of *topological arbiters* was studied in [Freedman and Krushkal 2012]. A topological arbiter is an invariant which assigns either 0 or 1 to each side A, B of any decomposition, subject to certain natural axioms. In particular, it is a topological invariant, and the values associated to the two sides of a decomposition are required to be different. The notion of a topological arbiter, together with an additional "Bing doubling" property, axiomatize the properties that an obstruction to the A-B slice problem should satisfy. [Freedman and Krushkal 2012] proved the existence of an uncountable collection of topological arbiters in dimension 4, but the existence of one also satisfying the Bing-doubling axiom remains an open question.

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Slava Krushkal

Please provide email

Department of Mathematics
University of Virginia
Charlottesville, VA 22904-4137
USA