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In the centuries since the discovery of Newton equation, the quest to solve the many-body problem has been one of the most persistent endeavours of mathematics and physics. Although progress was made in approximating it when the number of particles is small, a solution for large numbers of particles in any useful form seems simply impossible. The fundamental observation of Boltzmann was that the typical behavior for classical Hamiltonian systems in equilibrium is governed by ensemble averages (Gibbs states, in today's language). This avoided the difficulty of directly solving the Newton equations by postulating statistical ensembles, and led to modern statistical physics and ergodic theory. Boltzmann's formulation concerned systems in equilibrium; in other words, behavior of systems as the time approaches infinity.

At the other end is the kinetic theory for short-time behavior. Classical dynamics are exactly solvable when there is no interaction among particles, that is, in the case of free dynamics. For short time, classical dynamics can be understood by supplementing the free dynamics with collisions. The fundamental observation of kinetic theory — the idealization of the collision processes — is again due to Boltzmann in his celebrated work on the Boltzmann equation.

For systems neither in equilibrium nor near free dynamics (that is, for time scales too short for equilibrium theory but too long for kinetic theory) the most useful descriptions are still the classical macroscopic equations, for example, the Euler and Navier–Stokes equations. These are continuum formulations of conservation of mass and momentum, and also contain some phenomenological concepts such as viscosity. They are equations for macroscopic quantities such as density, velocity (momentum) and energy, while the Boltzmann equation is an equation for the probability density of finding a particle at a fixed position and velocity. The classical Hamiltonian plays no active role in either formulation, and all the microscopic effects are summarized by the viscosity in the Navier–Stokes equations, or the collision operator in the Boltzmann equation. However, the central theoretical question, that is, understanding the connection between large particle systems and their continuum approximations, remained unsolved and is still one of the fundamental questions in nonequilibrium statistical physics.

Since classical dynamics of large systems are all but impossible to solve, a more feasible goal is to replace the classical dynamics with stochastic dynamics. From the 1960s to early 1980s, tremendous effort was made by Dobrushin, Lebowitz, Spohn, Presutti, Spitzer, Liggett, and others to understand large stochastic particle systems. A key focus was to derive rigorously the classical phenomenological equations from the interacting particle systems in suitable scaling limits. The methods at the time were based on coupling and perturbative arguments; the systems which can be treated rigorously were restricted

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to special one-dimensional systems, perturbations of the symmetric simple exclusion or mean-field type interactions.

Together with Guo and Papanicoloau, Varadhan [1988] introduced the first general approach, the entropy method. The key ideas are the dissipative nature of the entropy, and large deviations. As long as the equilibrium measures of the dynamics are known, and the scaling is diffusive, this approach is very effective and now has been applied to many systems. We will now sketch the approach in [Guo et al. 1988]. The method is most transparent in a model (the hydrodynamical limit of this model was first proved under some more restrictive assumptions in [Fritz 1987]) where the particle number is replaced by a real-valued scalar field $\phi_x \in \mathbb{R}$, $x \in \{1, \cdots, N\}$, with periodic boundary condition so that $N+1 = 1$. These evolve as interacting diffusions. Denote by μ_0 the product measure such that the law of ϕ_x is given by $e^{-V(\phi_x)}$. Let $f_t\mu_0$ be the distribution of the field ϕ at the time t. The dynamics of ϕ will be given by the evolution equation for f_t,

$$\partial_t f_t = L f_t, \tag{1}$$

$$L = \sum_{j=1}^{N}\Big(\frac{\partial}{\partial\phi_j} - \frac{\partial}{\partial\phi_{j+1}}\Big)^2 - \sum_{j=1}^{N}\Big(V'(\phi_j) - V'(\phi_{j+1})\Big)\Big(\frac{\partial}{\partial\phi_j} - \frac{\partial}{\partial\phi_{j+1}}\Big).$$

This dynamics is reversible with respect to the invariant measure μ_0, and the Dirichlet form is given by

$$D(f) := -\int f\,Lf\,d\mu_0 = \sum_{j=1}^{N}\int\Big(\frac{\partial f}{\partial\phi_j} - \frac{\partial f}{\partial\phi_{j+1}}\Big)^2 d\mu_0 \tag{2}$$

We rescale the time diffusively so that the evolution equation becomes

$$\partial_t f_t = \varepsilon^{-2} L f_t, \tag{3}$$

where $\varepsilon = N^{-1}$ is the scaling parameter.

The dynamics (1) can be written as a conservation law,

$$d\phi_i = (w_{i+1} - w_i)\,dt + dM_i,$$
$$w_i = N^2 V'(\phi_i) - N^2 V'(\phi_{i-1})$$

where M_i are martingales and w_i is the microscopic current. The current is itself a gradient, and our dynamics is formally diffusive, that is, for any smooth test function J,

$$dN^{-1}\sum_i J(\varepsilon i)\phi_i \sim N^{-1}\sum_i J''(\varepsilon i)V'(\phi_i)\,dt + dM. \tag{4}$$

Denote by $\rho(x,t)$ the local average of ϕ around $x = \varepsilon i$,

$$\rho(x,t) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\delta+1}\sum_{|j-i|\le\delta\varepsilon^{-1}}\phi_j. \tag{5}$$

Let $p(\lambda)$ denote the pressure

$$p(\lambda) = \log\int_{\mathbb{R}} d\phi\, e^{\lambda\phi - V(\phi)}, \tag{6}$$

and $h(m)$ denote the free energy, that is, the Legendre transform of p. It is not hard to check that the martingale term in (4) vanishes in the limit, and hence that the main task for establishing the hydrodynamical limit of the form

$$\partial_t \rho = \partial_{xx} h'(\rho)$$

is to prove that we can replace $V'(\phi_i)$ in (4) by $h'(\rho(x,t))$, in the sense of a law of large numbers with respect to the distribution f_t satisfying (3).

Consider the local Gibbs measure with a chemical potential λ,

$$d\mu_\lambda(\phi_1, \cdots, \phi_N) = \exp\Big[\sum_j \lambda(\varepsilon j)\phi_j\Big] d\mu_0$$

where the chemical potential λ is allowed to depend on the site slowly. If f_t is a local Gibbs state, then certainly we can replace $V'(\phi_i)$ in (4) by $h'(\rho(x,t))$ in the sense of a law of large numbers. The key observation of [Guo et al. 1988] is to consider the evolution of the entropy,

$$\partial_t S(f_t) = -D(\sqrt{f_t}), \qquad S(f) = \int f \log f \, d\mu_0.$$

For a typical system of N variables given by a density f with respect to μ_0, the entropy $S(f)$ is of order N. This implies that

$$\int_0^t D(\sqrt{f_s})\, ds \le CN \tag{7}$$

This information alone is sufficient to establish that the solution of (3) is close enough to a local Gibbs state that the law of large numbers continues to hold.

Systems where the current is itself a gradient of some other function are known as *gradient systems*. For such systems, [Guo et al. 1988] provides a general framework to establish the hydrodynamical limit. However, many systems are of nongradient type. A simple illustrative example is to modify the dynamics (1) so that the generator L becomes the symmetric operator with Dirichlet form

$$D(f) := -\int f\, Lf\, d\mu_0 = \sum_{j=1}^N \int a(\phi_j, \phi_{j+1}) \Big(\frac{\partial f}{\partial \phi_j} - \frac{\partial f}{\partial \phi_{j+1}}\Big)^2 d\mu_0. \tag{8}$$

The current can be computed easily, and is given (up to scale factors) by

$$w_j = -\frac{\partial a(\phi_j, \phi_{j+1})}{\partial \phi_j} + \frac{\partial a(\phi_j, \phi_{j+1})}{\partial \phi_{j+1}} + a(\phi_j, \phi_{j+1})[\phi_1 - \phi_{j+1}]. \tag{9}$$

To establish the hydrodynamical limit, it is now required to prove that, when $x = \varepsilon j$, in some sense w_j can be replaced by

$$w_j = D(\rho(x,t))[\phi_{j+1} - \phi_j] \tag{10}$$

for some function D, which will be the diffusion coefficient of the hydrodynamical equation. The key observation of Varadhan's work on nongradient systems is that

$$w_j = D(\rho(x,t))[\phi_{j+1} - \phi_j] + LF \tag{11}$$

where F is some local function of ϕ_j and L is the generator given by (8). The idea is that functions of the type LF represent incoherent rapid fluctuations which vanish over the long time scale of the hydrodynamical limit. This fluctuation is indeed in the system, and the hydrodynamical limit can be established only if we properly account for its effect.

The sense in which (11) holds is the H_{-1} sense, corresponding to the vanishing of the variance in the central limit theorem for the corresponding additive functional. This goes back to Varadhan's earlier work on *tagged particles* [Kipnis and Varadhan 1986]. The problem of proving the convergence of tagged particles to appropriate diffusions is somewhat complementary to the hydrodynamical limit. Varadhan introduced the martingale method in this context so that the idea of viewing the system from the point of view of the particle can be implemented. These ideas have had broad influence not only in hydrodynamical limits, but also in homogenization theory and for random walk in random environment. In fact, there is even a more explicit connection between the tagged particle problems and the nongradient systems. Suppose that one gives each particle one of m different labels, and watches the evolution of the m different densities in the hydrodynamical limit. The corresponding particle systems are usually of nongradient form [Quastel 1992] as long as $m \geq 2$. This is a weak form of tagging, and the large-m limit of this system is a (weak) way to keep track of individual particles. It can be proved, via nongradient system methods, that each species of particles evolves according to a diffusion equation and, thus, the hydrodynamical limit of tagged particles in nonequilibrium is established [Quastel et al. 1999]. The advantage of this approach is that it can be done in nonequilibrium, identifying the collective drift imposed by the flow of the bulk towards equilibrium. However, it is strictly speaking not the behavior of a single tagged particle, but the average behavior of tagged particles with vanishing density.

The equation (11) is quite difficult to solve as it involves the full generator L. In order to solve it, Varadhan developed a method which can be viewed as an infinite-dimensional version of Hodge theory. This is a deep theory, and we shall only attempt to convey some of its flavor here. First note that, because of the entropy bound (7), one only has to solve (11) in equilibrium. So, the diffusion coefficient can be treated as a constant. For simplicity take $j = 0$. The current w_0 has the property that

$$\int w_0 \, d\mu_\lambda = 0 \tag{12}$$

for any Gibbs state with constant chemical potential λ. The space of functions with this property corresponds to a space of *closed forms*. A subspace of *exact forms* corresponds to the fluctuation terms LF. The deep result is that the exact forms are of codimension one in the space of closed forms with orthogonal complement corresponding to $\phi_{j+1} - \phi_j$; this solves (11). This approach, as it stands, is based on the integration-by-parts nature of L and applies only to reversible dynamics. It is possible to formulate it also for nonreversible dynamics, and the formal analogy between this equation and the Hodge theory can be strengthened [Varadhan and Yau 1997].

The two fundamental papers [Guo et al. 1988; Varadhan 1993] of Varadhan ushered in an era of hydrodynamical limits based on the idea of entropy. The developments following these two papers are astonishing, and we shall only mention a few. The approach of [Guo et al. 1988] was successfully applied to many systems, including interacting Brownian motions [Varadhan 1991], interacting Ornstein–Uhlenbeck processes [Olla and Varadhan 1991] and Ginzburg–Landau models [Rezakhanlou 1990]. The interacting

Brownian motions and interacting Ornstein–Uhlenbeck processes are continuum systems with no lattice structure. The hydrodynamical limit for the Ginzburg–Landau models was proved for all temperatures, including the phase transition region — a remarkable result. Furthermore, the approach of [Guo et al. 1988] was successfully applied to kinetic scaling, and led to the derivation of the Boltzmann equation from stochastic particle systems [Rezakhanlou 2004]. The idea that the solution of (3) is heuristically a local Gibbs states goes back many decades, but the estimates obtained in [Guo et al. 1988] are in fact strong enough to prove it. It is observed in [Yau 1991] that one can bypass many technical difficulties in [Guo et al. 1988] and prove directly that the local Gibbs states are in fact an approximate solution to (3) in the sense of relative entropy. The assumptions needed in this approach are (a) some ergodic properties of the dynamics, and (b) smoothness of solutions to the hydrodynamical equations. This method is more restrictive than [Guo et al. 1988] for diffusive systems, but it essentially relies only on the identification of the invariant measures of the dynamics, and it applies also to hyperbolic systems before the formation of shocks. It was adapted in [Olla et al. 1993] to derive the classical Euler equation from Hamiltonian systems with vanishing noise. This is the most significant advance since Morrey stated this problem in the 1960s. Once the hyperbolic equations develop shocks, a very different method is needed; see [Rezakhanlou 1991; Fritz and Tóth 2004] for references and related results.

Varadhan's work on nongradient systems requires a spectral gap of order ℓ^{-2} for the system in a box of side length ℓ. This inspired work on the estimates of spectral gaps of conservative dynamics, and it led to the development of martingale methods for estimating spectral gap for conservative dynamics. Using this spectral estimate, Varadhan and his coauthor [Varadhan and Yau 1997] established the hydrodynamical limit of lattice gas in the high-temperature phase.

The idea that the current can be decomposed into a dissipative term $\phi_j - \phi_{j+1}$ and a fluctuation term LF is a deep idea, and is really a rigorous statement of the so-called fluctuation-dissipation theorem from physics. In a sense, the insight that this equation is fundamental to the hydrodynamical limit is at least as significant as the solution of this equation for the specific model considered in [Varadhan 1993]. Although the fluctuation-dissipation equation was solved in [Varadhan 1993] only for reversible dynamics, it was realized that one can develop a method to solve this equation for nonreversible dynamics, provided that the spatial dimension is larger than two [Landim and Yau 1997]. This led to the derivation of the incompressible Navier–Stokes (INS) equations from stochastic lattice gases for dimension $d = 3$ [Quastel and Yau 1998]. The result obtained in [Quastel and Yau 1998] is very strong; it identifies the large deviation rate that the hydrodynamical equation is not a Leray solution, and does not assume that the INS equations have classical solutions. The physical significance is the following: The first principles equation governing a classical fluid is the Newton equation, which is time reversible and has no dissipation. The INS equations possess viscosity and are time irreversible. Therefore, a derivation of the INS equations from classical mechanics would have to answer the fundamental question relating to the origin of dissipation and the breaking of time reversibility in classical dynamics. Although the underlying dynamics in [Quastel and Yau 1998] is stochastic, it was proved that the viscosity in the INS equations was *strictly* larger than the original viscosity of the underlying stochastic dynamics. In other words, the deterministic part of the dynamics makes a nontrivial contribution to the viscosity. We remark that the condition $d = 3$ is critical. For dimension $d = 2$, it was proved that the hydrodynamical limit equations for such lattice gas models are not the INS equations. Indeed, even the diffusive scaling is incorrect, and there are logarithmic corrections. Although these works do not answer directly the fundamental question regarding the derivation of the incompressible Navier–Stokes equations from the classical dynamics, it is the first time we understand the generation of the viscosity from many particle dynamics. These developments are largely attributed to Varadhan's insight of the importance of the fluctuation-dissipation equations (11).

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