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FRAYDOUN REZAKHANLOU

LARGE DEVIATION AND HOMOGENIZATION

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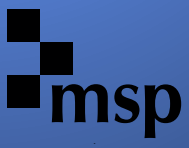

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LARGE DEVIATION AND HOMOGENIZATION

FRAYDOUN REZAKHANLOU

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The main theme of this expository article is to review some of Raghu Varadhan and his collaborators' contributions to the question of homogenization for the following stochastic models:

- stationary Hamilton–Jacobi (HJ) and Hamilon–Jacobi–Bellman (HJB) equation;
- random walk in random environment (RWRE);
- simple exclusion process (SEP).

All the above models share similar scaling behaviors and, in some sense, represent evolving height functions which are governed by local and random growth rules. In fact, the law of a RWRE satisfies an equation which resembles a discrete HJB equation, and the growth rates of the particle currents in SEP are described by a nonlinear function of the height differences. Reviewing Raghu Varadhan's fundamental contributions sheds light on some universal behavior of stochastic growth models.

The Hamilton–Jacobi and Hamilon–Jacobi–Bellman equations

To introduce the basic idea behind homogenization, we first consider the (inhomogeneous) Hamilton–Jacobi (HJ) equation,

$$u_t = H(x, u_x), \tag{1}$$

where H is stationary and ergodic in the first variable x. More precisely, we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F}$ a Borel σ-field on Ω and $\mathbb{P}$ a probability measure on $(\Omega, \mathcal{F})$, which is invariant with respect to a family of translation operators; that is, for every $x \in \mathbb{R}^d$, there exists a measurable function $\tau_x : \Omega \to \Omega$ so that $\tau_x \circ \tau_y = \tau_{x+y}$, and $\mathbb{P}(\tau_x A) = \mathbb{P}(A)$ for every $A \in \mathcal{F}$ and $x, y \in \mathbb{R}^d$. We also assume that τ_x is ergodic; that is, $\tau_x A = A$ for all $x \in \mathbb{R}^d$ implies that either $\mathbb{P}(A) = 1$ or 0.

Now, $H(x, p, \omega) = H_0(\tau_x \omega, p)$ where $H_0 : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a measurable function. We think of (x, t, u) as the *microscopic coordinates*, with the graph of $u(\cdot, t)$ representing a random interface. To switch to *macroscopic coordinates*, we set

$$u^\varepsilon(x, t; \omega) = \varepsilon u\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}; \omega\Big). \tag{2}$$

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We now have

$$u_t^\varepsilon = H\Big(\frac{x}{\varepsilon}, u_x^\varepsilon\Big). \tag{3}$$

We note that the right-hand side of (3) fluctuates greatly over macroscopic shifts in the position x. The huge fluctuation in H, though, does not necessarily imply correspondingly huge fluctuations in u^ε. This is the homogenization phenomenon; that is, we expect $u^\varepsilon \to \bar{u}$ as $\varepsilon \to 0$, with $\bar{u}$ solving a *homogenized HJ* equation

$$\bar{u}_t = \bar{H}(\bar{u}_x), \tag{4}$$

where $\bar{H} : \mathbb{R}^d \to \mathbb{R}$ is the *homogenized Hamiltonian* and does not depend on ω.

As our second example, we consider the Hamilton–Jacobi–Bellmann equation

$$u_t = H(x, u_x) + \tfrac{1}{2}\,\Delta\, u, \tag{5}$$

with $H(x, p) = H(x, p, \omega)$ as before. We define u^ε as in (2), and then (3) becomes

$$u_t^\varepsilon = H\Big(\frac{x}{\varepsilon}, u_x^\varepsilon\Big) + \frac{\varepsilon}{2}\,\Delta\, u^\varepsilon. \tag{6}$$

Again, we expect to have $u^\varepsilon \to \bar{u}$, with $\bar{u}$ satisfying an equation of the form (4) for a different homogenized Hamiltonian $\bar{H}$. Indeed, the homogenizations for both (3) and (6) have been achieved by Souganidis [1999], Rezakhanlou and Tarver [2000], Lions and Souganidis [2005], and Kosygina, Rezakhanlou and Varadhan [2006], provided that $H(x, p)$ is convex in p and satisfies suitable technical assumptions on which we do not elaborate here. (See also Kosygina and Varadhan [2008] when H is allowed to depend on the time variable.) Notably, [Kosygina et al. 2006] obtains a variational formula for $\bar{H}$. In the case of (6), $\bar{H}$ is given by

$$\bar{H}(p) = \inf_g \operatorname*{esssup}_\omega \Big[H_0\big(p + g(\omega), \omega\big) + \tfrac{1}{2}\nabla \cdot g(\omega) \Big] \tag{7}$$

where the essential supremum is taken with respect to the probability measure $\mathbb{P}$, and the infimum is taken over functions $g : \Omega \to \mathbb{R}^d$ such that $\mathbb{E}\, g = 0$ and $\nabla \cdot g = 0$ weakly. Here, ∇ is the generator of the group $\{\tau_x\}$; that is,

$$\nabla f(\omega) \cdot v = \lim_{t \to 0} \frac{1}{t}\big(f(\tau_{tv}\omega) - f(\omega)\big) \tag{8}$$

whenever the limit exists. We expect a similar formula to hold in the case of (3), namely,

$$\bar{H}(p) = \inf_g \operatorname*{esssup}_w \; H_0\big(p + g(\omega), \omega\big). \tag{9}$$

Before we turn to our next model, we make an observation regarding the homogenization of (6). Note that, if

$$H(x, p, \omega) = \tfrac{1}{2}|p|^2 + b(x, \omega) \cdot p + V(x, \omega) \tag{10}$$

and u is a solution of (5), then, by the Hopf–Cole transform, the function $w = e^u$ solves

$$w_t = \tfrac{1}{2}\,\Delta\, w + b(x, \omega) \cdot \nabla w + V(x, \omega)\, w(x, \omega). \tag{11}$$

By the Feynmann–Kac formula, there is a probabilistic representation for w using a diffusion with a drift b. More precisely, if $X(t, x; \omega)$ denotes the solution to

$$dX(t) = b(X(t), \omega)\, dt + d\beta(t), \qquad X(0) = x, \tag{12}$$

then

$$w(x, t; \omega) = E^{\omega} w(X(t, x; \omega), 0) \exp\Big(\int_0^t V\big(X(s, x; \omega), \omega\big)\, ds \Big). \tag{13}$$

Here, β is a standard Brownian motion, and E^{ω} denotes the expected value for the process $X(t)$. The function V is the potential and, if $V \le 0$, then $-V$ may be interpreted as a killing rate for the diffusion X. With this interpretation, $w(x, t; \omega)$ is the expected value of $w(\hat{X}(t), 0)$, with $\hat{X}$ denoting the diffusion with the killing. We now would like to use our probabilistic representation to rewrite u^{ε}. If

$$u^{\varepsilon}(x, 0; \omega) = f(x) \tag{14}$$

for a deterministic initial condition f, then

$$u^{\varepsilon}(x, t; \omega) = \varepsilon \log E^{\omega} \exp\Big[\varepsilon^{-1} f(\varepsilon X(t/\varepsilon, x/\varepsilon; \omega)) + \int_0^{t/\varepsilon} V(X(s, x/\varepsilon; \omega), \omega)\, ds \Big] \tag{15}$$

In particular,

$$u^{\varepsilon}(0, 1; \omega) = \varepsilon \log E^{\omega} \exp\Big[\varepsilon^{-1} f(X(\varepsilon^{-1}; \omega)) + \int_0^{\varepsilon^{-1}} V(X(s; \omega), \omega)\, ds \Big] \tag{16}$$

where $X(s; \omega) := X(s, 0; \omega)$ is the diffusion starting from the origin. On the other hand, since $\bar{H}$ is convex (which is evident from (7)) we may use the Hope–Lax–Oleinik formula to write

$$\bar{u}(x, t) = \sup_y \Big(f(y) - t \bar{L}\Big(\frac{y - x}{t} \Big) \Big), \tag{17}$$

where $\bar{L}$ is the convex conjugate of $\bar{H}$. In particular,

$$\lim_{\varepsilon \to 0} u^{\varepsilon}(0, 1; \omega) = \bar{u}(0, 1) = \sup_y \big(f(y) - \bar{L}(y) \big). \tag{18}$$

By a celebrated lemma of Varadhan, (18) is equivalent to saying that, for almost all ω, the diffusion $\hat{X}$ satisfies a *large-deviation principle* with rate function $\bar{L}$. When $b \equiv 0$ and

$$-V(x, \omega) = \sum_{j \in I} V_0(x - x_j), \tag{19}$$

with $\omega = \{x_j : j \in I\}$ being a Poisson point process and V_0 a continuous function of compact support, the large-deviation principle for $\hat{X}$ was earlier established by Sznitman [1998].

In words, the large-deviation principle for the diffusion $\hat{X}(\,\cdot\,; \omega)$ is equivalent to homogenization for the equation (6). Write P^{ω} for the law of the process $\hat{X}(\,\cdot\,; \omega)$. What we have in (18) is an example of a *quenched* large-deviation principle. We may also consider the annealed law

$$\bar{P} = \int P^{\omega}\, \mathbb{P}(d\omega) \tag{20}$$

and wonder whether an annealed large-deviation principle is true for the process $\hat{X}$. More precisely, whether or not

$$\lim_{\varepsilon\to 0} \varepsilon \log \int E^{\omega} \exp\Big[\varepsilon^{-1} f(X(\varepsilon^{-1},\omega)) + \int_0^{\varepsilon^{-1}} V(X(s,\omega),\omega)\, ds\Big]\, \mathbb{P}(d\omega) = \sup_y \big(f(y) - J(y)\big) \tag{21}$$

for a suitable rate function J. In terms of u^{ε}, this is equivalent to saying

$$\lim_{\varepsilon\to 0} \varepsilon \log \int e^{\varepsilon^{-1} u^{\varepsilon}(0,1;\omega)}\, \mathbb{P}(d\omega) = \sup_y \big(f(y) - J(y)\big). \tag{22}$$

This would follow if we can establish a large-deviation principle for the convergence of u^{ε} to $\bar{u}$. That is, if we can find a function $K_f(y;x,t)$ such that

$$\lim_{\varepsilon\to 0} \varepsilon \log \int e^{\varepsilon^{-1}\lambda u^{\varepsilon}(x,t;\omega)}\, \mathbb{P}(d\omega) = \sup_y \big(\lambda y - K_f(y;x,t)\big). \tag{23}$$

The annealed large deviation (22) in the case $b \equiv 0$ and V from (19) can be found in the manuscript [Souganidis 1999], but (23) remains open even when $b = 0$.

It is worth mentioning that there is also a variational description for the large-deviation rate function $\bar{L}$, namely

$$\bar{L}(v) = \inf_a \inf_{\mu\in\Gamma_{a,v}} \int L_0(\omega, a(\omega))\, \mu(d\omega), \tag{24}$$

where $L_0(\omega, v)$ is the convex conjugate of $H_0(\omega, p)$ and $\Gamma_{a,v}$ is the set of invariant measures for the diffusions

$$\mathcal{A}_a = a(\omega)\cdot\nabla + \tfrac{1}{2}\Delta \quad \text{with} \quad \int a(\omega)\, \mu(d\omega) = v.$$

In the case of (3), the generator $\mathcal{A}_a$ takes the form $a\cdot\nabla$ and, when H is periodic in x (that is, when Ω is the d-dimensional torus with $\mathbb{P}$ being the uniform measure), the formula (24) is equivalent to a formula of Mather for the averaged Lagrangian and our homogenization is closely related to the weak KAM theory. See Fathi and Maderna [2007] and Evans and Gomez [2001] for more details.

The random walk in a random environment

As our second class of examples, we consider a discrete version of the diffusion (12). This is simply a random walk in a random environment (RWRE). To this end, let us write $\mathcal{P}$ for the space of probability densities on the d-dimensional lattice $\mathbb{Z}^d$; that is, $p \in \mathcal{P}$ if $p : \mathbb{Z}^d \to [0,1]$ with $\sum_z p(z) = 1$. We set $\Omega = \mathcal{P}^{\mathbb{Z}^d}$, and $\omega \in \Omega$ is written as

$$\omega = (p_a : a \in \mathbb{Z}^d).$$

Given $\omega \in \Omega$, we write $X(n, a; \omega)$ to denote a random walk at time n with starting point $a \in \mathbb{Z}^d$ and transition probabilities p_a, $a \in \mathbb{Z}^d$. More precisely,

$$P^{\omega}\big(X(n+1) = y \;\big|\; X(n) = x\big) = p_x(y - x).$$

Given a function $g : \mathbb{Z}^d \to \mathbb{R}$, we write

$$T_n g(x) = E^\omega g(X(n, x; \omega)),$$

so that

$$T_1 g(x) = \sum_{y \in \mathbb{Z}^d} g(y)\, p_x(y - x).$$

To compare with (11) in the case $V \equiv 0$, we also write

$$w(x, n) = T_n g(x)$$

for a given initial g. This trivially solves

$$w(x, n + 1) = \big(T_1 w(\cdot\,, n)\big)(x).$$

To compare with (5), we set $u = \log w$ so that

$$u(x, n + 1) - u(x, n) = \big(Au(\cdot\,, n)\big)(x)$$

where

$$Ag(x) = \log T_1 e^g(x) - g(x) = \log \sum_z e^{g(x+z)-g(x)}\, p_x(z).$$

Now, homogenization means that we are interested in

$$\bar{u}(x, t) = \lim_{\varepsilon \to 0} \varepsilon u\Big(\Big[\frac{x}{\varepsilon}\Big], \Big[\frac{t}{\varepsilon}\Big]; \omega\Big),$$

provided that ω is distributed according to an ergodic stationary probability measure $\mathbb{P}$, where

$$\tau_x \omega = (p_{y+x} : y \in \mathbb{Z}^d).$$

(Here, $[a]$ denotes the integer part of a.)

Again, $\bar{u}$ solves (4) provided that $\lim_{\varepsilon \to 0} u^\varepsilon(x, 0) = \bar{u}(x, 0) = f(x)$ exists initially. The function $\bar{L}$ (the convex conjugate of $\bar{H}$) is the quenched large-deviation rate function for $X(n; \omega)$. More precisely, for any bounded continuous $f : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{n \to \infty} n^{-1} \log E e^{f(n^{-1} X(n, 0; \omega))} = \sup_y \big(f(y) - \bar{L}(y)\big).$$

This has been established under an ellipticity condition on p_x by Varadhan [2004]. See Bolthausen and Sznitman [2002] for a survey on earlier results. The analog of (7) is the following formula of Rosenbluth [2006]:

$$\bar{H}(p) = \inf_g \operatorname*{esssup}_\omega \sum_y p_0(y)\, e^{p \cdot z + g(\omega, z)} \tag{25}$$

with infimum over functions $\big(g(\cdot\,, z) : \Omega \to \mathbb{R} : z \in \mathbb{Z}^d\big)$ such that $\mathbb{E} g(\cdot\,, z) = 0$ and g is a "closed 1-form". By the latter we mean that, for every loop $x_0, x_1, \ldots, x_{k-1}, x_k = x_0$, we have that $\sum_{r=0}^{k-1} g(\tau_{x_r} \omega, x_{k+1} - x_k) = 0$.

We now turn to the annealed large deviations for a RWRE. For this, we need to select a tractable law for the environment. Pick a probability measure β on $\mathcal{P}$ and set $\mathbb{P}$ to be the product of β to obtain a law on $\mathcal{P}^{\mathbb{Z}^d}$. The annealed measure $\bar{P} = \int P^\omega \, \mathbb{P}(d\omega)$ has a simple description. For this, we write $Z(n) = X(n+1) - X(n)$ for the jump the walk performs at time n. We also define

$$N_{x,z}(n) = \#\big\{i \in \{0, 1, 2, \dots, n\} : X(i) = x, Z(i) = z\big\}.$$

We certainly have

$$P\big(X(1;\omega) = x_1, \dots, X(n,\omega) = x_n\big) = \prod_{z,x\in\mathbb{Z}^d} (p_x(z))^{N_{x,z}(n)},$$

$$\bar{P}\big(X(1) = x_1, \dots, X(n) = x_n\big) = \prod_{x\in\mathbb{Z}^d} \int \prod_z (p(z))^{N_{x,z}(n)} \beta(dp),$$

where now

$$N_{x,z}(n) = \#\big\{i \in \{0, 1, \dots, n-1\} : x_i = x, x_{i+1} - x_i = z\big\}.$$

Evidently, $\bar{P}$ is the law of a non-Markovian walk in $\mathbb{Z}^d$. Varadhan [2004] established the annealed large-deviations principle under a suitable ellipticity condition on β. The method relies on the fact that the environment seen from the walker is a Markov process for which Donsker–Varadhan theory may apply if we have enough control on the transition probabilities.

If we set

$$\begin{aligned} W_n &= \big(0 - X(n), X(1) - X(n), \dots, X(n-1) - X(n), X(n) - X(n)\big) \\ &= (s_{-n}, \dots, s_{-1}, s_0 = 0) \end{aligned}$$

for the chain seen from the location $X(n)$, then we obtain a walk of length n that ends at 0. The space of such walks is denoted by $\mathbf{W}_n$. Under the law $\bar{P}$, the sequence $W_1, W_2, \dots$ is a Markov chain with the following rule:

$$\bar{P}(W_{n+1} = T_z W_n \mid W_n) = \frac{\bar{P}(T_z W_n)}{\bar{P}(W_n)} = \frac{\int_{\mathcal{P}} p(z) \prod_a p(a)^{N_{0,a}} \beta(dp)}{\int_{\mathcal{P}} \prod_a p(a)^{N_{0,a}} \beta(dp)}, \tag{26}$$

where $N_{0,a} = N_{0,a}(W_n)$ is the number of jumps of size a from 0 for the walk W_n. Here, $T_z W_n$ denotes a walk of size $n+1$ which is formed by translating the walk W_n by $-z$ so that it ends at $-z$ instead of 0, and then making the new jump of size z so that it ends at 0.

We wish to establish a large-deviation principle for the Markov chain with transition probability $q(W, z)$ given by (26), where $W = W_n \in \bigcup_{m=0}^{\infty} \mathbf{W}_m$ and z is the jump size. We assume that, with probability one, the support of $p_0(\cdot)$ is contained in the set $D = \{z : |z| \le C_0\}$. Naturally, q extends to those infinite walks $W \in \mathbf{W}_\infty$ with $N_{0,a} < \infty$ for every $a \in D$. If we let $\mathbf{W}_\infty^{\mathrm{tr}}$ denote the set of transient walks, then the expression $q(W, z) = \mathbf{q}(W, T_z W)$ given by (26) defines the transition probability for a Markov chain in $\mathbf{W}_\infty^{\mathrm{tr}}$. Donsker–Varadhan theory suggests that the empirical measure

$$\frac{1}{n} \sum_{m=0}^{n-1} \delta_{W_m}$$

satisfies a large-deviation principle with a rate function

$$\mathcal{I}(\mu) = \int_{\mathbf{W}_\infty^{\mathrm{tr}}} \mathbf{q}_\mu(W, z) \log \frac{\mathbf{q}_\mu(W, z)}{\mathbf{q}(W, z)} \mu(dW),$$

where μ is any T-invariant measure on $\mathbf{W}_\infty^{\mathrm{tr}}$, and $\mathbf{q}_\mu(W, z)$ is the conditional probability of a jump of size z, given the past history. We then use the contraction principle to come up with a candidate for the large-deviation rate function

$$H(v) = \inf\Big\{ I(\mu) : \int z_0 \mu(dW) = v \Big\},$$

where $(z_j : j \in \mathbb{Z})$ denotes the jumps of a walk W. Several technical difficulties arise as one tries to apply Donsker–Varadhan theory, because of the non-compactness of the state space and the fact that the transition probabilities are not continuous. These issues are handled masterfully in [Varadhan 2004].

The simple exclusion process

We now turn to our final model. This time, our environment $\omega = (p_i(t) : i \in \mathbb{Z})$ is a collection of independent Poisson clocks. More precisely, p_i with $i \in \mathbb{Z}$ are independent, and each p_i is a Poisson process of rate 1; $p_i(t) = k$ for $t \in [\tau_1^i + \cdots + \tau_k^i, \tau_1^i + \cdots + \tau_{k+1}^i)$ with τ_j^i independent mean-1 exponential random variables. Given a realization of ω and an initial height function

$$h^0 \in \Gamma = \big\{ h : \mathbb{Z} \to \mathbb{Z} \mid 0 \le h(i+1) - h(i) \le 1 \big\},$$

we construct $h(i, t) = h(i, t; \omega)$ such that $h(\cdot, t; \omega) \in \Gamma$ for all t. More precisely, at each Poisson time $t = \tau_1^i + \cdots + \tau_k^i$, the height $h(i, t)$ increases by one unit provided that the resulting height function h^i belongs to Γ; otherwise, the increase is suppressed.

The process $h(\cdot, t)$ is a Markov process with the rule $h \to h^i$ with rate $\eta(i+1)\,(1 - \eta(i))$, where $\eta(i) = h(i) - h(i-1)$. The process $(\eta(i, t; \omega) : i \in \mathbb{Z})$ is also Markovian, with the interpretation that $\eta(i, t) = 1$ if the site i is occupied by a particle, and $\eta(i, t) = 0$ if the site i is vacant. Now, the growth $h \to h^i$ is equivalent to jumping a particle from site $i+1$ to i, provided that the site i is vacant. Since $h \in \Gamma$ is nondecreasing, we may define its inverse $x \in \Gamma'$, where

$$\Gamma' = \big\{ x : \mathbb{Z} \to \mathbb{Z} \mid x(h+1) > x(h) \big\}.$$

Since h increases at a site $i+1$ if the site i is occupied by a particle, we may regard $x(h)$ as the position of a particle of label h. Equivalently, we may interpret $h(i)$ as the label of a particle at an occupied site i.

The process $x(h, t; \omega)$ is also a Markov process with the rule $x(h, t) \to x(h, t) - 1$ with rate $\mathbb{1}\big(x(h, t) - x(h-1, t) > 1\big)$. In words, $x(h)$ decreases by one unit with rate 1, provided that the resulting configuration x_h is still in Γ'. For the construction of $x(h, t; \omega)$ we may use the clocks ω or, equivalently, we may use clocks that are assigned to sites $h \in \mathbb{Z}$. More precisely, if $\omega' = (p'_h(t) : h \in \mathbb{Z})$ is a collection of independent Poisson processes of rate 1, then we decrease $x(h)$ by one unit when the clock p'_h rings. The processes $x(h, t; \omega)$ and $x(h, t; \omega')$ have the same distribution. If we define $\zeta(h, t) = x(h, t) - x(h-1, t) - 1$, then $\zeta(h, t)$ represents the gap between the h-th and $(h-1)$-th particles in the exclusion process. The process

$(\zeta(h,t) : h \in \mathbb{Z})$ is the celebrated *zero-range process* and can be regarded as the occupation number at site h. The ζ-process is also Markovian, where a ζ-particle at site h jumps to site $h+1$ with rate $\mathbb{1}(\zeta(h) > 0)$.

As in the previous sections, we set

$$u^\varepsilon(x,t;\omega) = \varepsilon h\Big(\Big[\frac{x}{\varepsilon}\Big], \frac{t}{\varepsilon}; \omega\Big),$$
$$x^\varepsilon(u,t;\omega) = \varepsilon x\Big(\Big[\frac{u}{\varepsilon}\Big], \frac{t}{\varepsilon}; \omega\Big),$$

and as a homogenization we expect to have $u^\varepsilon \to \bar{u}$, $x^\varepsilon \to \bar{x}$, with $\bar{u}$ and $\bar{x}$ deterministic solutions to Hamilton–Jacobi equations

$$\bar{u}_t = \bar{H}_1(\bar{u}_x) = \bar{u}_x(1-\bar{u}_x), \tag{27}$$

$$\bar{x}_t = \bar{H}_2(\bar{x}_u) = \frac{1}{\bar{x}_u} - 1. \tag{28}$$

(See [Rezakhanlou 1991].) As for the large deviations, we will be interested in

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(u^\varepsilon(x,t) \geq u) = \lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(x^\varepsilon(u,t) \geq x) =: -W(x,u,t). \tag{29}$$

Evidently, $W(x,u,t) = 0$ if $u \leq \bar{u}(x,t)$ or $x \leq \bar{x}(u,t)$. However, we have that $W(x,u,t) > 0$ whenever $u > \bar{u}(x,t)$ or $x > \bar{x}(u,t)$. As it turns out,

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(u^\varepsilon(x,t) \leq u) = -\infty \tag{30}$$

for $u < \bar{u}(x,t)$ because, for such a number u,

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}(u^\varepsilon(x,t) \leq u) > 0,$$

as was demonstrated by Jensen and Varadhan [$\geq$ 2012] (see also [Jensen 2000] and [Varadhan 2004]). Quoting from [Jensen and Varadhan $\geq$ 2012], the statement (29) has to do with the fact that one may slow down $x(h,t)$ for $h \leq h_0$ in a time interval of order $O(\varepsilon^{-1})$ by simply slowing down $x(h_0,t)$. This can be achieved for an entropy price of order $O(\varepsilon^{-1})$. However, for $x^\varepsilon(u,t) \leq \bar{x}(u,t) - \delta$, with $\delta > 0$, we need to speed up $O(\varepsilon^{-1})$-many particles for a time interval of order $O(\varepsilon^{-1})$. This requires an entropy price of order $O(\varepsilon^{-2})$.

As was observed by Seppäläinen [1999], both the h and x processes enjoy a strong monotonicity property. More precisely, if we write $x(h,t;\omega) = T_t^\omega x^0(h)$ for the x-process starting from the initial configuration $x^0 \in \Gamma'$, then $T_t^\omega(\sup_\alpha x_\alpha^0) = \sup_\alpha T_t^\omega(x_\alpha^0)$. In words, if the initial height $x^0 = \sup_\alpha x_\alpha^0$ is the supremum of a family of height functions x_α^0, then it suffices to evolve each x_α^0 separately for a given realization of ω, and take the supremum afterwards. From this, it is not hard to show that such a strong monotonicity must be valid for W and this, in turn, implies that W solves a HJ equation of the form

$$W_t = K(W_x, W_u). \tag{31}$$

Here, the initial data $W(x,u,0)$ is the large-deviation rate function at initial time. Of course we assume that there is a large-deviation rate function initially, and would like to derive a large-deviation principle at later times. In the case of the exclusion or zero-range process, it is not hard to guess what K is,

because, when the process is at equilibrium, the height function at a given site has a simple description. To construct the equilibrium measures for the x-process, we pick a number $b \in (0,1)$ and define a random initial-height function $x(\cdot\,,0)$ by the requirement that $x(\cdot\,,0)=0$ and that $(x(h+1,0)-x(h,0)-1 : h\in\mathbb{Z})$ are independent geometric random variables of parameter b. That is, $x(h+1,0)-x(h,0)=k+1$ with probability $(1-b)b^k$. Let us write $\mathbb{P}^b$ for the law of the corresponding process $x(h,t;\omega)$. Using Cramer's large-deviation theorem, we can readily calculate that, for u positive,

$$W(x,u,0) = -\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}^b(x^\varepsilon(u,0)\ge x) = u\Big(I_1\Big(\frac{x}{u}+1,b\Big)\Big)^+ \tag{32}$$

where $I_1(r,b) = r\log[r/(b(1+r))] - \log[(1-b)(1+r)]$. As is well known (see for example Chapter VIII, Corollary 4.9 of Liggett [2005]), $-x(0,t)$ is a Poisson process which decreases one unit with rate b. Again Cramer's theorem yields

$$W(x,0,t) = bt\Big(I_2\Big(\frac{-x}{bt}\Big)\Big)^+ \tag{33}$$

where $I_2(r) = r\log r - r + 1$. The expressions (31)–(33) provide us with enough information to figure out what K is. We refer to [Seppäläinen 1998] for a large-deviation principle of the form (29) for a related particle system known as Hammersley's model.

Alternatively, we may study the large deviation of the particle densities. For this purpose, we define the empirical measure by

$$\pi^\varepsilon(t,dx) = \pi^\varepsilon(t,dx;\omega) = \varepsilon\sum_i \delta_{\varepsilon i}(dx)\,\eta(i,t/\varepsilon;\omega).$$

We regard π^ε as an element of the Skorohod space $X = \mathcal{D}([0,T];M)$, where M is the space of locally bounded measures. The law $\omega\mapsto\pi^\varepsilon(t,dx;\omega)$ induces a probability measure $\mathcal{P}^\varepsilon$ on X.

The hydrodynamic limit for the exclusion process means that $\mathcal{P}^\varepsilon\to\mathcal{P}$ where $\mathcal{P}$ is concentrated on the single *entropy* solution of

$$\bar\rho_t = (\bar\rho(1-\bar\rho))_x \tag{34}$$

for a given initial data $\bar\rho(x,0)=\bar\rho^0(x)$. The function $\hat\rho$ is related to the macroscopic height function $\bar u$ by $\bar\rho = \bar u_x$. In [Jensen and Varadhan $\ge$ 2012], a large-deviation principle has been established for the convergence of $\mathcal{P}^\varepsilon$. Roughly,

$$\mathcal{P}^\varepsilon\big(\pi^\varepsilon(t,dx) \text{ is near } \mu(t,dx)\big) \approx e^{-\varepsilon^{-1}\mathcal{I}(\mu)} \tag{35}$$

with the following rate function $\mathcal{I}$: First, $\mathcal{I}(\mu)=+\infty$ unless $\mu(t,dx)=m(x,t)\,dx$ and m is a weak solution of (34). However, when $0<\mathcal{I}(m)<\infty$, then m is a non-entropic solution of (34). In fact $\mathcal{I}(\mu) = \mathcal{I}_0(\mu)+\mathcal{I}_{\mathrm{dyn}}(\mu)$, where $\mathcal{I}_0(\mu)$ is the large-deviation rate function coming from the initial deviation and depends only on our choice of initial configurations, and $\mathcal{I}_{\mathrm{dyn}}(\mu)$ is the contribution coming from dynamics and quantitatively measures how the entropy condition is violated. By "entropy condition" we mean that, for a pair (φ,q) with φ convex and $\varphi'\bar H_1' = q'$ for $\bar H_1(p)=p(1-p)$, we have

$$\varphi(\bar\rho)_t + q(\bar\rho)_x \le 0 \tag{36}$$

in the weak sense. The left-hand side is a negative distribution, which can only be a negative measure. As our discussions around (31) and (32) indicate, the invariant measures play an essential role in determining the large-deviations rate function. As it turns out, the relevant φ to choose is simply the large-deviation rate function for the invariant measure, which is given by

$$\varphi(m) = m \log m + (1-m)\log(1-m) + \log 2.$$

Here, for the invariant measure we choose a Bernoulli measure ν under which $(\eta(i) : i \in \mathbb{Z})$ are independent and $\nu(\eta(i) = 1) = 1/2$. To measure the failure of the entropy solution, we take a weak solution m for which the corresponding

$$\varphi(m)_t + q(p)_x = \gamma \ = \gamma^+ - \gamma^-$$

is a measure, with γ^+ and γ^- representing the positive and negative part of γ. We now have

$$\mathcal{I}_{\mathrm{dyn}}(\mu) = \gamma^+(\mathbb{R} \times [0, T]).$$

It is customary in equilibrium statistical mechanics to represent a state as a probability measure with density $(1/Z)e^{-\beta H}$, with H some type of energy and Z the normalizing constant. In non-equilibrium statistical mechanics, a large-deviation principle of the form (35) offers an analogous expression, with $\mathcal{I}(\mu)$ playing the role of "effective" energy (or, rather, potential). What we learn from [Jensen and Varadhan ≥ 2012] is that, after the entropy solution, the most frequently visited configurations are those associated with non-entropic solutions, and the entropic price for such visits is measured by the amount the inequality (36) fails. Even though the entropy solutions for scalar conservation laws are rather well understood, our understanding of non-entropic solutions is rather poor, perhaps because we had no reason to pay attention to them before. The remarkable work [Jensen and Varadhan ≥ 2012] urges us to look more deeply into non-entropic solutions for gaining insight into the way the microscopic densities deviate from the solution to the macroscopic equations.

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Fraydoun Rezakhanlou

rezakhan@math.berkeley.edu

Department of Mathematics
University of California
Berkeley CA 94720
USA