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Naresh Jain

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NARESH JAIN

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Professor Kai Lai Chung's contributions to probability theory have had a major influence on several areas of research in the subject. I will restrict my comments to some of his work in two areas, sums of independent random variables and the theory of Markov chains, which led to a significant amount of further work, including some of my own.

Sums of independent random variables

Kai Lai has made many outstanding contributions to this field, but I would like to concentrate on his paper [1948]. If $X_1, X_2, \ldots$ is a sequence of real-valued independent random variables, and

$$S_n = X_1 + X_2 + \cdots + X_n \quad \text{for } n \geq 1$$

denotes the sequence of partial sums, then the almost-sure behavior of "large values" of $\{S_n\}$ was very well understood. Indeed, in the independent and identically distributed (i.i.d) case, Hartman and Wintner in [1941] had already proved their celebrated law of the iterated logarithm: $EX_1 = 0$ and $EX_1^2 = 1$ imply

$$\limsup_{n\to\infty} \frac{S_n}{(2n \log\log n)^{1/2}} = 1 \quad \text{a.s.} \tag{1}$$

In a more general non-i.d. context, Feller in [1943] had written almost the final word. Chung [1948] observed that, if S_n is replaced by $|S_n|$ in (1), the assertion remains valid. However, if S_n^* denotes $\max_{1\leq j\leq n} |S_j|$, then the behavior of "small values" of S_n^* had yet to be understood. He studied this problem in [1948] including the non-i.d. situation and proved that, denoting $E(S_n^2)$ by s_n^2, if $EX_j \equiv 0$ then, under a natural third moment assumption,

$$\liminf_{n\to\infty} \frac{S_n^*}{s_n(\log\log s_n)^{-1/2}} = 8^{-1/2}\pi \quad \text{a.s.} \tag{2}$$

To prove this result, he obtained the very profound probability estimate

$$P(S_n^* < cs_n) = \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \exp\Big(-\frac{(2j+1)^2\pi^2}{8c^2}\Big) + O\big((\log\log s_n/\log s_n)^{1/2}\big). \tag{3}$$

This contains the probability distribution for a standard one-dimensional Brownian motion process $\{B_t : t \geq 0\}$, if S_n^*/s_n is replaced by $\max_{0\leq t\leq 1} |B_t|$ on the left side, and the second term is replaced by zero on the right side.

In the i.i.d. case, two questions arose after Chung's work. The first one was raised by Chung himself: If $EX_1 = 0$ and $EX_1^2 = 1$, does (2) hold without any further assumptions, with $s_n^2 = n$? The second natural question was to obtain the analogue of (2) if X_1 is in the domain of attraction of a stable law.

As to the first question, several papers appeared on the subject getting close to the conditions stipulated by Chung. The question was finally settled in the affirmative by Jain and Pruitt in [1975]. The probability distribution given in (3) by Chung played a key role in the final solution. For the second question, if the index of stability is $\alpha < 2$, it was not even clear if one should expect an analogue of (2).

Fristedt had already observed that an analogue of (1) could not exist if $\alpha < 2$. However, Fristedt and Pruitt [1971] and Jain and Pruitt [1973] showed, under different conditions on X_1, the existence of a real sequence $\{b_n\}$ increasing to infinity such that $\liminf(S_n^*/b_n) = c$ a.s., with $0 < c < \infty$. However, it was not clear if the constant c depended on the distribution of X_1 or on the limit distribution alone.

Donsker and Varadhan [1977] approached these problems through their large-deviations probability estimates for stable processes, and obtained explicit expressions for the limit constants. Jain [1982] was then able to show, through an invariance principle, that the limit constant for the lim inf behavior of (S_n^*/b_n) is the same as for the relevant stable process obtained by Donsker and Varadhan [1977].

The story by no means ends here. For a two-parameter Brownian motion $B(s,t)$ with $0 \leq s, t \leq 1$, the leading term of the "small ball" probability estimate for

$$P\Big(\max_{0\leq s,t\leq 1} |B(s,t)| \leq c\Big) \quad \text{as } c \downarrow 0$$

is of great interest, and turned out to be a challenging problem, solved by Bass [1988] and Talagrand [1994]. Much work has been done by other authors for parameter dimension larger than 2; these results, however, are not as definitive as in the two-dimensional parameter case. Many difficult questions still remain to be answered, and we can expect these investigations to continue, all owing their origins to Chung's pioneering work.

Markov chains

It is difficult to imagine that anybody working in the area of Markov processes would not be familiar with Chung's monograph, *Markov Chains with Stationary Transition Probabilities* [1967]. This monograph deals with countable-state Markov chains in both discrete time (Part I) and continuous time (Part II). Much of Kai Lai's fundamental work in the field is included in this monograph. My comments will be confined to Part I. Here, for the first time, Kai Lai gave a systematic exposition of the subject, which includes classification of states, ratio ergodic theorems, and limit theorems for functionals of the chain.

For a general state space, Doeblin had given a classification scheme in a seminal paper in [1937]. This and other work of Doeblin had a major impact on the field, and led to further developments by Chung [1964], Doob [1953], Harris [1956], and Orey [1959; 1971]. In the early 1960s there were a number of basic ingredients of a general state-space theory that would lead to an exact counterpart of Part I of

Chung [1967]. Much fundamental ground work, including positive recurrence (the so-called Doeblin's condition), was done by Doob [1953]. Harris [1956] introduced his recurrence condition: There exists a nonzero σ-finite measure φ on the state space S such that $\varphi(E) > 0$ implies that starting from every $x \in S$, E is visited infinitely often a.s. He proved the existence of a (unique) σ-finite invariant measure π under this condition. If $\pi(S) = +\infty$, one could call the process null-recurrent, and one could ask if $\pi(E) < \infty$ implied that, for every $x \in S$, $P^n(x, E) \to 0$ as $n \to \infty$; here, $P^n(x, E)$ denotes the n-step transition probability from x to E. This result was conjectured by Orey [1959], and was a natural extension to the general state-space situation of the corresponding well-known result for a countable state chain. Under Harris's recurrence condition, one could also ask if an analogous ratio ergodic theorem was true; namely, if $\pi(F) > 0$ and $\pi(E) < \infty$, does

$$\sum_{j=1}^{n} P^j(x, E) \Big/ \sum_{j=1}^{n} P^j(y, F) \to \pi(E)/\pi(F) \quad \text{as } n \to \infty \tag{4}$$

for *all* $x, y \in S$? These questions were answered in [Jain 1966] under Chung's guidance. Chung gave an example, reported in [Jain 1966], to show that in the general case (as opposed to the countable state space case) (4) is true only for π-almost all x, y, and not for *all* x, y. The situation is different when the state space is not countable, because one could stay in a π-null set for a rather long time! A little later, Jain and Jamison [1967] introduced an irreducibility condition: There exists a nonzero σ-finite measure φ on S such that $\varphi(E) > 0$ implies that, starting from every $x \in S$, the process visits E with positive probability. In this paper they essentially brought the program of Doeblin [1937] and Chung [1964] to completion. Chung's influence can be seen throughout these works. For other work in the area one can refer to monographs by Orey [1971], Revuz [1984] and Meyn and Tweedie [1993].

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NARESH JAIN: *School of Mathematics, University of Minnesota, Minneapolis, MN 55455, United States*