CELEBRATIO MATHEMATICA

Kai Lai Chung

MICHAEL CRANSTON

CONDITIONAL BROWNIAN MOTION AND CONDITIONAL GAUGE

2012

msp

http://celebratio.org/

CELEBRATIO MATHEMATICA
Kai Lai Chung (2012)

CONDITIONAL BROWNIAN MOTION AND CONDITIONAL GAUGE

MICHAEL CRANSTON

First published: 2008

Through his works and words, Kai Lai Chung has been the spur for substantial developments in the understanding of conditional Brownian motion and its application to the theory of Schrödinger operators. Many in the field received mail or phone calls from Chung with interesting and provocative questions on the subject. At the Spring 1982 meeting of the Seminar on Stochastic Processes, he posed an interesting question on the lifetime of conditional Brownian motion. The resolution of this question (described below) has led to wide-ranging developments. His foundational work with Rao on the gauge theorem, to mention just one of his many works in this area, has served as the motivation for many developments in the understanding of Schrödinger operators and their semigroups. And, of course, *From Brownian Motion to Schrödinger's Equation* [1995] with Zhongxin Zhao has served as guide to developments in the field. In this short, semi-accurate, historical note, I'd like to outline a few developments that trace their origins to the encouragement of Chung. I'd like to apologize in advance for the many works which are not mentioned here, due in large part to an interest in brevity.

First, an introduction is in order. Let B denote Brownian motion on $\mathbb{R}^d$ defined on the probability space $(\Omega, \mathcal{F}_t, \{P_x\}_{x\in\mathbb{R}^d})$. For $D \subset \mathbb{R}^d$, let $p(t, x, y)$ be its transition density when killed at time

$$\tau_D = \inf\{t > 0 : B_t \notin D\}$$

(the heat kernel on D.) Given a positive super-harmonic function h on D, define

$$p^h(t, x, y) = \frac{p(t, x, y)\, h(y)}{h(x)}.$$

This is the transition density for a new diffusion, called the h-process or conditional Brownian motion. We denote by P_x^h the measure on path-space corresponding to Brownian motion started at x and with transition density $p^h(t, x, y)$. In the case when h is the Martin kernel with pole at the Martin boundary point ξ, then the conditional Brownian motion exits the domain at the boundary point ξ, in the sense that B_t converges P_x^h-a.s. to ξ in the Martin topology as t approaches the path lifetime, τ_D. If $h(\cdot) = G_D(\cdot, y)$, where G_D is the Green function for D and $y \in D$, then the h-process will converge to y as t approaches the path lifetime. When h is the Martin kernel with pole at ξ, we denote the resulting measure by P_x^ξ and, when $h(\cdot) = G(\cdot, y)$, by P_x^y. These were developments due to Doob [1957] in his study of probabilistic

First published in *Selected works of Kai Lai Chung*, edited by F. AitSahlia, E. Hsu, and R. Williams, MR 2841270, Zbl 1165.60302.

versions of the Fatou boundary-limit results for harmonic functions. Now it's been known for some time that, if D is bounded, then, for unconditioned Brownian motion,

$$E_x[\tau_D] < c_d \operatorname{vol}(D)^{2/d}, \tag{1}$$

and, if λ_1 is the first Dirichlet eigenvalue for $\frac{1}{2}\Delta$ on D, then

$$\lim_{t\to\infty} \frac{1}{t} \log P_x(\tau_D > t) = -\lambda_1. \tag{2}$$

Using the Martin boundary, denoted here by $\partial_M D$, the expected lifetime can be expressed as

$$E_x[\tau_D] = \int_{\partial_M D} E_x^{\xi}[\tau_D]\, \omega_x(d\xi),$$

where ω_x is the exit distribution of Brownian motion on $\partial_M D$, also known as the harmonic measure. So, by Fubini, $E_x^{\xi}[\tau_D]$ is finite ω_x-almost surely. Chung's question is this:

When is $E_x^{\xi}[\tau_D]$ bounded uniformly in x and ξ?

Or, more generally, when is $E_x^{\xi}[\tau_D]$ finite? This innocuous-sounding question turned out to have quite broad implications. It led to the introduction of some very interesting ideas from analysis into probability theory, such as the boundary Harnack principle, Whitney chains, Littlewood–Paley g-function and intrinsic ultracontractivity.

The first result on this question, due to McConnell and the author [1983], was that there is a positive constant c so that, if $D \subset \mathbb{R}^2$ and h is a positive harmonic function on D, then

$$E_x^h[\tau_D] \le c \operatorname{vol}(D). \tag{3}$$

This is the analog then of (1) in $d = 2$. An example was given of a bounded $D \subset \mathbb{R}^3$ with a $\xi \in \partial_M D$ for which $E_x^{\xi}[\tau_D] = \infty$. Thus, the analog of (1) cannot hold for $d > 2$ without further assumptions. First, a word or two on the proof of (3). This relies on decomposing the domain D into subregions by means of the 2^m-level sets of the function h. That is,

$$D = \bigcup_{m=-\infty}^{\infty} D_m \quad \text{where} \quad D_m = \{x \in D : 2^{m-1} < h(x) < 2^{m+1}\}.$$

The conditional Brownian motion viewed at the successive hitting times to

$$C_m = \{x \in D : h(x) = 2^m\}$$

forms a birth and death Markov chain on $\{2^m : m \in \mathbb{Z}\}$, with probability 2/3 of going up and 1/3 of going down. This implies that the number of visits to the C_m are geometrically distributed random variables. These have finite expectation with a value independent of m. The other key observation is that the expected amount of time the conditional Brownian motion spends in D_m starting from C_m is equivalent (since $1/2 \le h(y)/h(x) \le 2$ for $x \in C_m$, $y \in D_m$) to the amount of time standard Brownian motion spends in D_m starting from C_m. Combining this observation with (1) gives that the expected time spent in D_m starting on C_m by the h-process is bounded by $C_d\,\mathrm{vol}(D_m)^{2/d}$. Using the strong Markov property and summing leads to an upper bound of

$$E_x^h[\tau_D] \le C_d \sum_{m=-\infty}^{\infty} \mathrm{vol}(D_m)^{2/d}.$$

In case $d = 2$, the sum is bounded by $2\,\mathrm{vol}(D)$, leading to the result that there is a constant c such that (3) holds for $D \subset \mathbb{R}^2$, $x \in D$, and h any positive superharmonic on D. Since $2/d < 1$ for $d \ge 3$, the finiteness of $\sum_{m=-\infty}^{\infty} \mathrm{vol}(D_m)^{2/d}$ does not generally hold, and leads to interesting questions about the influence of the regularity of the boundary and its effect on the size of the sets D_m. (The relation between boundary regularity and the growth of harmonic functions is a key issue in the subject.) This question was addressed by Bañuelos [1987], Falkner [1987], Bass and Burdzy [1992], DeBlassie [1988], Kenig and Pipher [1989], and myself [1985], among others in the higher-dimensional case. The results of Bañuelos [1987] incorporated many of the types of domains encountered in analysis, namely Lipschitz, NTA (non-tangentially accessible), John- and BMO-extension (uniform) domains. In order to describe the results in [Bañuelos 1987], we consider a Whitney decomposition of D. This is a collection of closed squares Q_j with sides parallel to the coordinate axes and $D = \bigcup_j Q_j$ with the properties

$$\begin{aligned} &Q_j^o \cap Q_k^o = \varnothing \quad \text{if } j \ne k,\\ &\frac{1}{4} \le \frac{l(Q_j)}{l(Q_k)} \le 4 \quad \text{if } Q_j \cap Q_k \ne \varnothing,\\ &1 \le \frac{d(Q_j, \partial D)}{l(Q_j)} \le 4\sqrt{d} \quad \text{for all } j. \end{aligned}$$

A Whitney chain connecting Q_j and Q_k is a sequence of Whitney squares $\{Q_{m_i}\}_{i=0}^{n}$ with $Q_{m_0} = Q_j$, $Q_{m_n} = Q_k$ and $Q_{m_i} \cap Q_{m_{i+1}} \ne \varnothing$. An important fact about Whitney squares is that there is a positive constant c so that, for any positive harmonic function h in D and adjacent Whitney squares $Q_j \cap Q_k \ne \varnothing$, we have $h(x) < c\,h(y)$ for $x \in Q_j$, $y \in Q_k$. Whitney chains are very well suited to the study of conditional Brownian motion. The reason is that, due to Harnack's inequality, any positive harmonic function will be "flat" on the Whitney square Q_j. This means that the transition densities $p^h(t, x, y)$ and $p(t, x, y)$ will be equivalent on Q_j, which means the behavior of ordinary and conditional Brownian motion will be comparable on Q_j. Now, define the quasi-hyperbolic distance from $x \in Q_j$ to x_0 by first setting $d(x) = \mathrm{dist}(x, \partial D)$ and then putting,

$$\rho_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} \frac{ds}{d(\gamma(s))},$$

with the inf being taken over all rectifiable curves in D from x_1 to x_2. Taking points $x_1 \in Q_j$, $x_2 \in Q_k$ we have

$$\rho_D(x_1, x_2) \approx \text{length of shortest Whitney chain from } Q_j \text{ to } Q_k.$$

Note that repeated applications of Harnack's inequality in successive squares in a Whitney chain implies that $h(x_1) \le c^{\rho_D(x_1,x_2)} h(x_2)$. If we fix an $x_0 \in D$ and write $\rho_D(x) = \rho_D(x, x_0)$, this implies that, for some constant C,

$$D_m \subset \{x \in D : \rho_D(x) > C|m|\}.$$

Now, a result of Smith and Stegenga [1990] implies that, for a class $H(0)$ of domains, called Hölder of order 0 (which includes Lipschitz, NTA, John- and BMO-extension domains), one has $\rho_D(\,\cdot\,, x_0) \in L^p(D)$ for any $0 < p < \infty$. Using this, Bañuelos obtains for D a bounded Hölder of order 0 domain so that

$$\sum_{m=-\infty}^{\infty} \operatorname{vol}(D_m)^{2/d} < \infty.$$

This implies that $H(0)$ domains are regular enough so that an analog of (3) holds for them in all dimensions. There are also beautiful connections in simply connected planar domains between the behavior of conditional Brownian motion and the hyperbolic geometry of the region. This was developed in Bañuelos and Carrol [1993], and Davis [1957]. We start our exposition of this connection with an observation of Bañuelos [1992]. If D is a simply connected planar domain, and $\varphi : B(0, 1) \to D$ maps the unit disc $B(0, 1)$ of the complex plane conformally onto D with $\varphi(0) = x$, then

$$g_*^2(\varphi)(\theta) = \frac{1}{\pi} \int_{B(0,1)} \log\Big(\frac{1}{|z|}\Big) \frac{1 - |z|^2}{|z - e^{i\theta}|^2} \, |\varphi'(z)|^2 \, dz$$

is the Littlewood–Paley square function. Recalling that the Green function of $B(0, 1)$ with pole at the origin is $\log(1/|z|)$, and that the Green function is preserved by conformal mappings, it is easy to deduce that, for h a positive harmonic function on D with the representation

$$h(z) = \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|} \, d\mu(\theta)$$

with μ a positive Borel measure on $\partial B(0, 1)$, that

$$\begin{aligned} E_x^h[\tau_D] &= \frac{1}{h(x)} \int_D G_D(x, y)\, h(y)\, dy \\ &= \frac{1}{h(x)} \int_{B(0,1)} \log\Big(\frac{1}{|z|}\Big)\, h(\varphi(z))\, |\varphi'(z)|^2 \, dx\, dy \\ &= \frac{1}{h(x)} \int_0^{2\pi} g_*^2(\varphi)(\theta)\, d\mu(\theta). \end{aligned} \tag{4}$$

Since $\mu([0, 2\pi]) = h(x)$ and

$$g_*^2(\varphi)(\theta) \le C \int_{B(0,1)} |\varphi'(z)|^2 \, dz \le C \operatorname{vol}(D),$$

it follows that

$$E_x^h[\tau_D] \le C \operatorname{vol}(D),$$

thus giving another derivation of the lifetime estimate in the special case of simply connected planar domains. But this gives additional information, as developed in Bañuelos and Carrol [1993]. There, the authors observed that, if $K(z, \xi)$ is the Poisson kernel for $B(0, 1)$ with pole at $\xi \in \partial B(0, 1)$, then there are positive constants c and C such that

$$c \sup_{\varphi} g_*^2(\varphi)(0) \;\le\; \sup_{\varphi} \int_{B(0,1)} K(z, 1) K(z, -1)\, |\varphi'(z)|^2 \, dx\, dy \;\le\; C \sup_{\varphi} g_*^2(\varphi)(0),$$

where the sup is taken over all conformal mappings $\varphi : B(0,1) \to D$ with $\varphi(0) = x$. But another equivalence holds for $K(z,1)K(z,-1)$. Namely, if $d(z,\Gamma)$ denotes the hyperbolic distance in $B(0,1)$ from z to the geodesic $\tilde{\Gamma} = [-1,1]$, then

$$\tfrac{1}{4}K(z,1)K(z,-1) \le e^{-2d(z,\tilde{\Gamma})} \le K(z,1)K(z,-1).$$

Using the conformal invariance of the hyperbolic metric, writing d_D for the hyperbolic metric in D, and putting these two equivalences together yields the existence of two positive constants c and C such that

$$c \sup_{\Gamma} \int_D e^{-2d_D(z,\Gamma)} \le \sup_{x,h} E_x^h[\tau_D] \le C \sup_{\Gamma} \int_D e^{-2d_D(z,\Gamma)}.$$

This has a beautiful corollary involving the Whitney decomposition mentioned above. Let Q be a Whitney cube with center z_Q, and let T_Q be the total amount of time spent in Q before τ_D. Then, for Martin boundary points ξ_1, ξ_2 and for Γ the hyperbolic geodesic connecting them, there are positive constants c and C such that

$$\tfrac{1}{4}e^{-Cd_D(z_Q,\Gamma)} \le E_{\xi_1}^{\xi_2}[T_Q] \le e^{-cd_D(z_Q,\Gamma)}.$$

This is a quantitative statement about how closely the conditional Brownian motion from ξ_1 to ξ_2 follows the hyperbolic geodesic from ξ_1 to ξ_2. Davis [1988] pursued this connection further in estimating the variance of τ_D under the measure $E_{\xi_1}^{\xi_2}$. If Q and R are Whitney squares, then setting

$$P_Q = P_{\xi_1}^{\xi_2}(\tau_{D\cap Q^c} < \tau_D) \quad \text{and} \quad P_R = P_{\xi_1}^{\xi_2}(\tau_{D\cap R^c} < \tau_D)$$

and letting $\delta(D)$ be the area of the largest disc which can be inscribed in D yields

$$|\operatorname{Cov}_{\xi_1}^{\xi_2}(T_Q, T_R)| \le Ce^{-c\delta_D(z_Q,z_R)} \operatorname{vol}(Q)\operatorname{vol}(R)\,(P_Q + P_R) \quad \text{and}$$
$$\operatorname{Var}_{\xi_1}^{\xi_2}(\tau_D) \le \delta(D)\, E_{\xi_1}^{\xi_2}[\tau_D].$$

The first of these shows exactly how the decay of the dependence between the occupation times T_Q and T_R depends on the hyperbolic distance between Q and R. The second confirms the intuition that the conditional Brownian motion speeds up when traversing narrow channels. (If D is a rectangle of length n and width $1/n$, then, for ξ_1 and ξ_2 on opposite ends of the long side of the rectangle,

$$E_{\xi_1}^{\xi_2}[\tau_D] \le c \quad \text{and} \quad \operatorname{Var}_{\xi_1}^{\xi_2}[\tau_D] \le c/n.$$

Thus, the conditional motion must go a distance n in a time with bounded expectation, independent of n, but with variance bounded by $1/n$. This means the mass of the measure $P_{\xi_1}^{\xi_2}$ is concentrating on paths which make the length-n trip in a time which is some constant that doesn't depend on n.)

Refinements and further progress in these directions can be found in the works of Griffin, McConnell and Verchota [1993a], Griffin, Verchota and Vogel [1993b], Zhang [1996], Davis and Zhang [1994], and Xu [1991], to name but a few.

Now, let's turn our attention to the problem of deciding to what extent the analog of (2) holds for conditional Brownian motion. From the case of a ball $D = \{x : |x| < r\}$ in Euclidean space where

$$P_0(\tau_D > t) = P_0^{\xi}(\tau_D > t)$$

for every boundary point ξ, one might suspect that, with some smoothness in $d > 2$ and maybe even with $\mathrm{vol}(D) < \infty$ in $d = 2$, if $H^+(D)$ is the class of positive harmonic functions on D, then

$$\lim_{t\to\infty} \frac{1}{t} \log P_x^h(\tau_D > t) = -\lambda_1 \quad \text{for } x \in D \text{ and } h \in H^+(D). \tag{5}$$

This was addressed in De Blassie [1988] where it was proved that (5) holds provided D is a Lipschitz domain with sufficiently small Lipschitz constant. Later, Kenig and Pipher [1989] extended this result to Lipschitz domains and NTA domains. Perhaps the nicest approach is due to Bañuelos [1991] and Bañuelos and Davis [1989], which illuminates the relation between the tail behavior of the lifetime of conditional Brownian motion and intrinsic ultracontractivity. The notion of intrinsic ultracontractivity is defined in Davies and Simon [1986] as the property that the semigroup of the ground-state transformation of an operator maps L^2 to L^∞. To make this definition precise in the current setting, if φ_1 is the first Dirichlet eigenfunction for $\frac{1}{2}\Delta$ on D, define a semigroup on $L^2(\varphi_1^2\,dx)$ by

$$P_t^{\varphi_1} f(x) = \int_D \frac{e^{\lambda_1 t} p(t,x,y)}{\varphi_1(x)\varphi_1(y)} f(y)\,\varphi_1^2(y)\,dy \quad \text{for } f \in L^2(\varphi_1^2\,dx).$$

Then, the domain D is defined to be intrinsically ultracontractive (IU) if there exist constants C_t such that

$$|P_t^{\varphi_1} f(x)| \le C_t \|f\|_{L^2(\varphi_1^2 dx)} \quad \text{for } t > 0.$$

An important consequence of IU is that for any $\varepsilon > 0$ there is a $t(\varepsilon)$ such that

$$(1-\varepsilon)\,e^{-\lambda_1 t}\varphi_1(x)\varphi_1(y) \le p(t,x,y) \le (1+\varepsilon)\,e^{-\lambda_1 t}\varphi_1(x)\varphi_1(y). \tag{6}$$

Since, for any $h \in H^+(D)$,

$$P_x^h(\tau_D \ge t) = \frac{1}{h(x)} \int_D p(t,x,y)h(y)\,dy \le 1,$$

it follows easily from (5) that

$$\lim_{t\to\infty} \frac{1}{t} \log P_x^h(\tau_D \ge t) = -\lambda_1,$$

giving the Bañuelos analog of (2) for conditional Brownian motion on IU domains. In the case of planar domains of finite area, Bañuelos and Davis [1989] proved the following analog of IU for each $x \in D$:

$$\lim_{t\to\infty} \frac{e^{\lambda_1 t} p(t,x,y)}{\varphi_1(x)\varphi_1(y)} = 1 \text{ uniformly in } y \in D.$$

This implies that the analog of (2) for conditional Brownian motion holds for planar domains of finite area.

Another application of conditional Brownian motion, which has been an area of research to which Professor Chung has made many contributions, is to the study of the Schrödinger equation by means of the Feynman–Kac formula. A seminal paper on the subject was that of Aizenman and Simon [1982], who used path-integral techniques (the Feynman–Kac formula) to prove Harnack's inequality for Schrödinger operators. Consider, with $d > 2$ for ease of presentation, a potential V satisfying

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|<r} \frac{|V(y)|}{|x-y|^{d-2}}\,dy = 0.$$

The class of such potentials is called the Kato class, and is denoted by K_d. They are particularly well suited to the Newtonian potential, and thus as well to the occupation properties of Brownian motion. Now, for $f \in C(\partial D)$, consider the Dirichlet problem

$$\begin{aligned} \tfrac{1}{2}\Delta u(x) + V(x)u(x) &= 0 \quad \text{for } x \in D, \\ u(x) &= f(x) \quad \text{for } x \in \partial D. \end{aligned} \tag{7}$$

The Gauge Theorem of Chung and Rao (see the article of Ruth Williams in this volume) says that either $E_x\big[e^{\int_0^{\tau_D} V(B_s)\,ds}\big] \equiv \infty$ on D, or this quantity (called the gauge) is bounded on D. Let's assume that the second alternative of this dichotomy holds. Then, by Feynman–Kac, the solution of (7) is given by

$$u(x) = E_x\big[e^{\int_0^{\tau_D} V(B_s)\,ds} f(B_{\tau_D})\big].$$

Let's suppose now that D is a Lipschitz domain so that the Euclidean and Martin boundary of D are the same. Decompose the Feynman–Kac formula using conditional Brownian motion,

$$u(x) = \int_{\partial D} E_x^y\big[e^{\int_0^{\tau_D} V(B_s)\,ds}\big] f(y)\, P_x(B_{\tau_D} \in dy). \tag{8}$$

The analog of Chung's question regarding the finiteness of the expected lifetime of conditional Brownian motion, as well as his question regarding the finiteness of the gauge, is:

$$\text{When is } E_x^y\big[e^{\int_0^{\tau_D} V(B_s)\,ds}\big] < \infty\,?$$

The quantity

$$u(x, y) = E_x^y\big[e^{\int_0^{\tau_D} V(B_s)\,ds}\big]$$

is known as the conditional gauge. Under the conditions set down above, namely that D be a Lipschitz domain and $V \in K_d$, a dichotomy (similar to the Gauge Theorem) holds: either

$$E_x^y\big[e^{\int_0^{\tau_D} V(B_s)\,ds}\big] \equiv \infty$$

or there are positive constants c and C such that

$$c \le E_x^y\big[e^{\int_0^{\tau_D} V(B_s)\,ds}\big] \le C \quad \text{for all } x, y \in D \cup \partial D. \tag{9}$$

This is called the Conditional Gauge Theorem (CGT). It can be viewed as a statement on the mixing properties of conditional Brownian motion. The potential V may possess singularities. The CGT says that these singularities can't be so bad that P_x^y-paths would miss them, in the sense that

$$E_x^y\big[e^{\int_0^{\tau_D} V(B_s)\,ds}\big] < \infty$$

for one pair of points x, y, but

$$E_z^w\big[e^{\int_0^{\tau_D} V(B_s)\,ds}\big] = \infty$$

for another pair z, w. That is, under both measures P_x^y and P_z^w, the occupation distributions of paths are similar enough that they will simultaneously give a finite answer or an infinite answer when asked about the value of the conditional gauge. This requires some smoothness of ∂D with its resulting effect on the behavior of the Green function. Early results on the subject were those of Falkner [1983] and Zhao [1983; 1984]. In the fundamental works of Zhao, the CGT was proved for Kato-class potentials on the ball, and then domains with C^2 boundary. For Lipschitz domains and Kato potentials, the result was proven in Cranston, Fabes and Zhao [1988]. The extension to Lipschitz domains of the CGT used the so-called $3G$-theorem. This result says that, if G is the Green function for $\frac{1}{2}\Delta$ on D, then there is a positive constant C such that

$$\frac{G(x,z)G(z,y)}{G(x,y)} \le C\Big(\frac{1}{|x-z|^{d-2}} + \frac{1}{|y-z|^{d-2}}\Big). \tag{10}$$

The left-hand side in (10) is the Green function for conditional Brownian motion started at x and conditioned to exit D at y. This is the occupation density for conditional Brownian motion in D, in the sense that the total expected amount of time spent by B in $A \subset D$ with respect to the measure P_x^y is

$$\int_A \frac{G(x,z)G(z,y)}{G(x,y)}\,dz.$$

The right-hand side of (10) is the sum of the Newtonian potentials with poles at x and y, respectively. These are the occupation densities for unconditioned Brownian motion in $\mathbb{R}^d$ started at x and y. The $3G$-inequality says that, if V is in K_d and thus well adapted to the occupation measure of (unconditioned) Brownian motion, then it is also well adapted to the occupation measure of conditional Brownian motion. In the case when the conditional gauge is finite, the CGT permits comparisons between potential theoretic quantities for the two operators $-\frac{1}{2}\Delta$ and $-\frac{1}{2}\Delta+V$. This lies close to the original motivation of Aizenman and Simon [1982]. For example, suppose that for some $f \in C(\partial D)$,

$$\begin{aligned} \tfrac{1}{2}\Delta v(x) &= 0 \quad \text{for } x \in D,\\ v(x) &= f(x) \quad \text{for } x \in \partial D. \end{aligned}$$

and

$$\begin{aligned} -\tfrac{1}{2}\Delta u(x) + V(x)u(x) &= 0 \quad \text{for } x \in D,\\ u(x) &= f(x) \quad \text{for } x \in \partial D. \end{aligned}$$

Then, $v(x) = E_x[f(B_{\tau_D})]$ and, since

$$c \le E_x^y[e^{\int_0^{\tau_D} V(B_s)\,ds}] \le C,$$

it follows from (4) that $cv(x) \le u(x) \le Cv(x),\ x \in D$. With this equivalence, Harnack's inequality, and even the boundary Harnack inequality, can be deduced for positive solutions in D of

$$-\tfrac{1}{2}\Delta u(x) + V(x)u(x) = 0.$$

Many other similar conclusions follow in an equally easy manner. Using the simple formula

$$u(x,y) = \frac{G_V(x,y)}{G(x,y)}, \tag{11}$$

it follows that

$$cG(x,y) \le G_V(x,y) \le CG(x,y) \quad \text{for } x, y \in D,$$

where G_V is the Green function for $-\frac{1}{2}\Delta + V$. Since the Martin kernels $K(x,\xi)$ and $K_V(x,\xi)$ are the limit of ratios of the Green functions $G(x,\xi)$ and $G_V(x,\xi)$, it follows as well that

$$cK(x,y) \le K_V(x,y) \le CK(x,y) \quad \text{for } x,y \in D,$$

where K and K_V are the Martin kernels for $\frac{1}{2}\Delta$ and $-\frac{1}{2}\Delta + V$, respectively. Two-dimensional versions of these results appeared in Bass and Burdzy [1995], Cranston [1989], McConnell [1990], and Zhao [1988]. Results similar in flavor and which also incorporate the notion of IU above are due to Bañuelos [1992], who proved that, when the conditional gauge is finite and D is a Lipschitz or NTA domain, there exist positive constants c_t and C_t such that

$$c_t p(t,x,y) \le p_V(t,x,y) \le C_t p(t,x,y) \quad \text{for } t > 0 \text{ and } x,y \in D,$$

where p_V is the heat kernel for $-\frac{1}{2}\Delta + V$. An additional result of Bañuelos in this connection is that, if the conditional gauge is finite and D is an $H(0)$ domain (as described earlier), then the operator $-\frac{1}{2}\Delta + V$ is IU. It's interesting to note that the proofs used log-Sobolev inequalities. Further developments appear in a series of papers by Chen and Song [2002; 2003], and Chen [2002], among others. In [Chen and Song 2002], the authors follow the developments of Bañuelos [1992], and consider the conditional gauge problem for the fractional Laplacian, $(-\Delta)^\alpha$ for $0 < \alpha < 2$, and potentials in the suitably modified Kato class $K_{\alpha,d}$, where $V \in K_{\alpha,d}$ if

$$\lim_{r\to 0} \sup_{\{x\in\mathbb{R}^d\}} \int_{|x-y|<r} \frac{|V(y)|}{|x-y|^{d-\alpha}}\, dy = 0.$$

In this paper, Chen and Song [2002] deduced the CGT on $C^{1,1}$ domains for the operator $(-\Delta)^\alpha$ and $K_{\alpha,d}$ potentials. The proper process to use in the Feynman–Kac representation in this case is the symmetric stable process of order α, X, rather than the Brownian motion used when considering Δ. Their approach was to split the potential, writing $V = V_1 + V_2$ for $V_2 \in \mathbf{L}^\infty$ and V_1 with a small Kato norm, that is, with small

$$\sup_{x\in D} \int_D \frac{|V_1(y)|}{|x-y|^{d-\alpha}} dy.$$

Then, by a simple lemma of Khasminski, they show that the Green functions $G^\alpha_{V_1}$ for $(-\Delta)^\alpha + V_1$ and G^α for $(-\Delta)^\alpha$ on D satisfy $G^\alpha_{V_1} \approx G^\alpha$, in the sense that there are positive constants c and C such that $cG^\alpha \le G^\alpha_{V_1} \le CG^\alpha$. This equivalence can then be used to prove that $(-\Delta)^\alpha + V$ is IU. However, $(-\Delta)^\alpha + V$ being IU implies that $G^\alpha \approx G^\alpha_V$. Now, using a formula analogue to (7), the finiteness of the right-hand side follows from the inequality $G^\alpha_{V_1} \le CG^\alpha$. From the 3$G$-Theorem for the Green function G^α on D, the CGT follows. This was extended in [Chen and Song 2003] to $H(0)$ domains, again by an approach inspired by [Bañuelos 1992].

Relations between subcriticality and boundedness of the conditional gauge have been investigated by Zhao [1992]. Consider the class

$$B_c = \{q : \mathbb{R}^d \to \mathbb{R} : \operatorname{supp} q \text{ is compact}\} \cap \mathbf{L}^\infty.$$

The operator $-\frac{1}{2}\Delta + V$ is called subcritical if

$$\text{for all } q \in B_c \text{ there exists } \varepsilon > 0 \text{ such that } -\tfrac{1}{2}\Delta + V + \varepsilon q \ge 0.$$

This amounts to a strict positivity of $-\frac{1}{2}\Delta + V$.

For a subclass of K_d potentials which satisfy a condition at ∞, Zhao [1992] proved that subcriticality is equivalent to

$$u(x, y) = E_x^y\big[e^{\int_0^\infty V(B_s)\,ds}\big] \text{ is bounded on } \mathbb{R}^d \times \mathbb{R}^d.$$

There were many other equivalences in that work which go a long way toward establishing the power of the approach in investigating the Schrödinger operator.

Generalizations of the Conditional Gauge Theorem to broader classes of Markov processes and potentials, including measures, have been carried out in Chen and Song [2002] and Chen [2002]. In the last work, Chen has proved gauge and conditional gauge theorems for a new class of Kato potentials, which even includes singular measures and general transient Borel right processes. And, most strikingly, following a suggestion of Chung, he proved that the CGT is actually the Gauge Theorem for the conditional process!

In this review, we've examined some of the many results which have connections with the works of Chung to be found in this volume. While we haven't explicitly drawn the connections, we hope that these ties will become obvious to any reader of this volume. Finally, the author would like to express his gratitude to Professor Chung for introducing him to the fascinating problems in this area.

First published in *Selected works of Kai Lai Chung*, edited by F. AitSahlia, E. Hsu, and R. Williams, MR 2841270, Zbl 1165.60302.

References

[Aizenman and Simon 1982] M. Aizenman and B. Simon, "Brownian motion and Harnack inequality for Schrödinger operators", *Comm. Pure Appl. Math.* **35**:2 (1982), 209–273. MR 84a:35062 Zbl 0459.60069

[Bañuelos 1987] R. Bañuelos, "On an estimate of Cranston and McConnell for elliptic diffusions in uniform domains", *Probab. Theory Related Fields* **76**:3 (1987), 311–323. MR 89c:60090 Zbl 0611.60071

[Bañuelos 1991] R. Bañuelos, "Intrinsic ultracontractivity and eigenfunction estimates for Schrödinger operators", *J. Funct. Anal.* **100**:1 (1991), 181–206. MR 92k:35066 Zbl 0766.47025

[Bañuelos 1992] R. Bañuelos, "Lifetime and heat kernel estimates in nonsmooth domains", pp. 37–48 in *Partial differential equations with minimal smoothness and applications* (Chicago, IL, 1990), IMA Vol. Math. Appl. **42**, Springer, New York, 1992. MR 93c:31005

[Bañuelos and Carroll 1993] R. Bañuelos and T. Carroll, "Conditioned Brownian motion and hyperbolic geodesics in simply connected domains", *Michigan Math. J.* **40**:2 (1993), 321–332. MR 94k:60116 Zbl 0805.60077

[Bañuelos and Davis 1989] R. Bañuelos and B. Davis, "Heat kernel, eigenfunctions, and conditioned Brownian motion in planar domains", *J. Funct. Anal.* **84**:1 (1989), 188–200. MR 91a:60203 Zbl 0676.60073

[Bass and Burdzy 1992] R. F. Bass and K. Burdzy, "Lifetimes of conditioned diffusions", *Probab. Theory Related Fields* **91**:3-4 (1992), 405–443. MR 93e:60155 Zbl 0739.60069

[Bass and Burdzy 1995] R. F. Bass and K. Burdzy, "Conditioned Brownian motion in planar domains", *Probab. Theory Related Fields* **101**:4 (1995), 479–493. MR 96b:60199 Zbl 0834.60089

[Chen 2002] Z.-Q. Chen, "Gaugeability and conditional gaugeability", *Trans. Amer. Math. Soc.* **354**:11 (2002), 4639–4679. MR 2003i:60127 Zbl 1006.60072

[Chen and Song 2002] Z.-Q. Chen and R. Song, "General gauge and conditional gauge theorems", *Ann. Probab.* **30**:3 (2002), 1313–1339. MR 2003f:60135 Zbl 1017.60086

[Chen and Song 2003] Z.-Q. Chen and R. Song, "Conditional gauge theorem for non-local Feynman–Kac transforms", *Probab. Theory Related Fields* **125**:1 (2003), 45–72. MR 2003m:60213 Zbl 1025.60032

[Chung and Zhao 1995] K. L. Chung and Z. X. Zhao, *From Brownian motion to Schrödinger's equation*, Grundlehren der Mathematischen Wissenschaften **312**, Springer, Berlin, 1995. MR 1329992 Zbl 0819.60068

[Cranston 1985] M. Cranston, "Lifetime of conditioned Brownian motion in Lipschitz domains", *Z. Wahrsch. Verw. Gebiete* **70**:3 (1985), 335–340. MR 87a:60088 Zbl 0581.60062

[Cranston 1989] M. Cranston, "Conditional Brownian motion, Whitney squares and the conditional gauge theorem", pp. 109–119 in *Seminar on stochastic processes, 1988* (Gainesville, FL, 1988), Progr. Probab. **17**, Birkhäuser, Boston, MA, 1989. MR 90j:60078 Zbl 0689.60077

[Cranston and McConnell 1983] M. Cranston and T. R. McConnell, "The lifetime of conditioned Brownian motion", *Z. Wahrsch. Verw. Gebiete* **65**:1 (1983), 1–11. MR 85d:60150 Zbl 0506.60071

[Cranston et al. 1988] M. Cranston, E. Fabes, and Z. Zhao, "Conditional gauge and potential theory for the Schrödinger operator", *Trans. Amer. Math. Soc.* **307**:1 (1988), 171–194. MR 90a:60135 Zbl 0652.60076

[Davies and Simon 1986] E. B. Davies and B. Simon, "L^1-properties of intrinsic Schrödinger semigroups", *J. Funct. Anal.* **65**:1 (1986), 126–146. MR 87f:35061 Zbl 0613.47039

[Davis 1988] B. Davis, "Conditioned Brownian motion in planar domains", *Duke Math. J.* **57**:2 (1988), 397–421. MR 89j:60112 Zbl 0663.60060

[Davis and Zhang 1994] B. Davis and B. Zhang, "Moments of the lifetime of conditioned Brownian motion in cones", *Proc. Amer. Math. Soc.* **121**:3 (1994), 925–929. MR 94i:60097 Zbl 0805.60079

[DeBlassie 1988] R. D. DeBlassie, "Doob's conditioned diffusions and their lifetimes", *Ann. Probab.* **16**:3 (1988), 1063–1083. MR 89i:60153

[Doob 1957] J. L. Doob, "Conditional Brownian motion and the boundary limits of harmonic functions", *Bull. Soc. Math. France* **85** (1957), 431–458. MR 22 #844 Zbl 0097.34004

[Falkner 1983] N. Falkner, "Feynman–Kac functionals and positive solutions of $\frac{1}{2}\Delta u + qu = 0$", *Z. Wahrsch. Verw. Gebiete* **65**:1 (1983), 19–33. MR 86m:60188a Zbl 0496.60078

[Falkner 1987] N. Falkner, "Conditional Brownian motion in rapidly exhaustible domains", *Ann. Probab.* **15**:4 (1987), 1501–1514. MR 88m:60195 Zbl 0627.60068

[Griffin et al. 1993a] P. S. Griffin, T. R. McConnell, and G. Verchota, "Conditioned Brownian motion in simply connected planar domains", *Ann. Inst. H. Poincaré Probab. Statist.* **29**:2 (1993), 229–249. MR 94g:60155 Zbl 0777.60073

[Griffin et al. 1993b] P. S. Griffin, G. C. Verchota, and A. L. Vogel, "Distortion of area and conditioned Brownian motion", *Probab. Theory Related Fields* **96**:3 (1993), 385–413. MR 94j:60149 Zbl 0794.60081

[Kenig and Pipher 1989] C. E. Kenig and J. Pipher, "The h-path distribution of the lifetime of conditioned Brownian motion for nonsmooth domains", *Probab. Theory Related Fields* **82**:4 (1989), 615–623. MR 91a:60211

[McConnell 1990] T. R. McConnell, "A conformal inequality related to the conditional gauge theorem", *Trans. Amer. Math. Soc.* **318**:2 (1990), 721–733. MR 90g:60072 Zbl 0705.60063

[Smith and Stegenga 1990] W. Smith and D. A. Stegenga, "Hölder domains and Poincaré domains", *Trans. Amer. Math. Soc.* **319**:1 (1990), 67–100. MR 90i:30012 Zbl 0707.46028

[Xu 1991] J. Xu, "The lifetime of conditioned Brownian motion in planar domains of infinite area", *Probab. Theory Related Fields* **87**:4 (1991), 469–487. MR 92d:60086 Zbl 0718.60092

[Zhang 1996] B. Zhang, "On the variances of occupation times of conditioned Brownian motion", *Trans. Amer. Math. Soc.* **348**:1 (1996), 173–185. MR 96e:60151 Zbl 0849.60078

[Zhao 1983] Z. X. Zhao, "Conditional gauge with unbounded potential", *Z. Wahrsch. Verw. Gebiete* **65**:1 (1983), 13–18. MR 86m:60188b Zbl 0521.60074

[Zhao 1984] Z. X. Zhao, "Uniform boundedness of conditional gauge and Schrödinger equations", *Comm. Math. Phys.* **93**:1 (1984), 19–31. MR 85i:35041 Zbl 0545.35087

[Zhao 1988] Z. Zhao, "Green functions and conditioned gauge theorem for a 2-dimensional domain", pp. 283–294 in *Seminar on stochastic processes, 1987* (Princeton, NJ,1987), Progr. Probab. Statist. **15**, Birkhäuser, Boston, MA, 1988. MR 91e:35079 Zbl 0667.35065

[Zhao 1992] Z. Zhao, "Subcriticality and gaugeability of the Schrödinger operator", *Trans. Amer. Math. Soc.* **334**:1 (1992), 75–96. MR 93a:81041 Zbl 0765.60063

MICHAEL CRANSTON: mcransto@math.uci.edu

Department of Mathematics, University of California, Irvine, Irvine CA 92697, United States