The Work of Vaughan F. R. Jones

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It gives me great pleasure that I have been asked to describe to you some of the very beautiful mathematics which resulted in the awarding of the Fields Medal to Vaughan F. R. Jones at ICM '90.

In 1984 Jones descovered an astonishing relationship between von Neumann algebras and geometric topology. As a result, he found a new polynomial invariant for knots and links in 3-space. His invariant had been missed completely by topologists, in spite of intense activity in closely related areas during the preceding 60 years, and it was a complete surprise. As time went on, it became clear that his discovery had to do in a bewildering variety of ways with widely separated areas of mathematics and physics, some of which are indicated in Figure 1. These included (in addition to knots and links) that part of statistical mechanics having to do with exactly solvable models, the very new area of quantum groups, and also Dynkin diagrams and the representation theory of simple Lie algebras. The central connecting link in all this mathematics was a tower of nested algebras which Jones had discovered some years earlier in the course of proving a theorem which is known as the "Index Theorem".

My plan is to begin by discussing the Index Theorem, and the tower of algebras which Jones constructed in the course of his proof. After that, I plan to return to the chart in Figure 1 in order to indicate how this tower of algebras served as a bridge between the diverse areas of mathematics which are shown on the chart. I will restrict my attention throughout to one very special example of the tower construction, and so also to one special example of the associated link invariants, in order to make it possible to survey a great deal of mathematics in a very short time. Even with the restriction to a single example, this is a very ambitious plan. On the other hand, it only begins to touch on Vaughan Jones' scholarly contributions.

1. The Index Theorem

Let $\mathbf{M}$ denote a von Neumann algebra. Thus $\mathbf{M}$ is an algebra of bounded operators acting on a Hilbert space $\mathscr{H}$. The algebra $\mathbf{M}$ is called a *factor* if its center consists only of scalar multiples of the identity. The factor is *type* II_1 if it admits a linear functional, called a trace, $\mathrm{tr} : \mathbf{M} \to \mathbb{C}$, which satisfies the following three conditions:

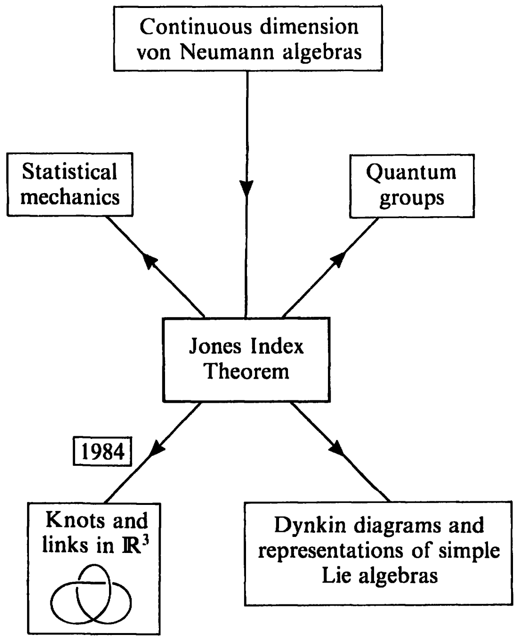


Fig. 1. The Jones Index Theorem

$$\begin{aligned}
\mathrm{tr}(\mathbf{xy}) &= \mathrm{tr}(\mathbf{yx}) \text{ for all } \mathbf{x},\mathbf{y} \in \mathbf{M}\\
\mathrm{tr}(\mathbf{1}) &= 1\,.\\
\mathrm{tr}(\mathbf{xx}^*) &> 0 \text{ for all } \mathbf{x} \in \mathbf{M}, \text{ where } \mathbf{x}^* \text{ is the adjoint of } \mathbf{x}.
\end{aligned}$$

In this situation it is known that the trace is unique, in the sense that it is the only linear form satisfying the first two conditions. An old discovery of Murray and von Neumann was that factors of type II_1 provide a type of "scale" by which one can measure the dimension $\dim_{\mathbf{M}}(\mathscr{H})$ of $\mathscr{H}$. The notion of dimension which occurs here generalizes the familiar notion of integer-valued dimensions, because for appropriate $\mathbf{M}$ and $\mathscr{H}$ it can be any non-negative real number or ∞.

The starting point of Jones' work was the following question: if $\mathbf{M}_1$ is a type II_1 factor and if $\mathbf{M}_0 \subset \mathbf{M}_1$ is a *subfactor*, is there any restriction on the real numbers which occur as the ratio

$$\lambda = \dim_{\mathbf{M}_0}(\mathscr{H}) / \dim_{\mathbf{M}_1}(\mathscr{H}) \ ?$$

The question has the flavor of questions one studies in Galois theory. On the face of it, there was no reason to think that λ could not take on any value in $[1,\infty]$, so Jones' answer came as a complete surprise. He called λ the *index* $[\mathbf{M}_1 : \mathbf{M}_0]$ of $\mathbf{M}_0$ in $\mathbf{M}_1$, and proved a type of rigidity theorem about type II_1 factors and their subfactors.

The Jones Index Theorem. *If* $\mathbf{M}_1$ *is a* II_1 *factor and* $\mathbf{M}_0$ *a subfactor, then the possible values of the index* $[\mathbf{M}_1 : \mathbf{M}_0]$ *are restricted to:*

$$[4,\infty] \cup \{4\cos^2(\pi/p), \quad \text{where } p \in \mathbb{N}, p \geq 3\}.$$

Moreover, each real number in the continuous part of the spectrum $[4,\infty]$ *and also in the discrete part* $\{4\cos^2(\pi/p),\ p \in \mathbb{N}, p \geq 3\}$ *is realized.*

We now sketch the idea of the proof, which is to be found in [Jo1]. Jones begins with the type II_1 factor $\mathbf{M}_1$ and the subfactor $\mathbf{M}_0$. There is also a tiny bit of additional structure: In this setting there exists a map $\mathbf{e}_1 : \mathbf{M}_1 \to \mathbf{M}_0$, known as the *conditional expectation* of $\mathbf{M}_1$ on $\mathbf{M}_0$. The map $\mathbf{e}_1$ is a *projection*, i.e. $(\mathbf{e}_1)^2 = \mathbf{e}_1$.

His first step is to prove that the ratio λ is independent of the choice of the Hilbert space $\mathscr{H}$. This allows him to choose an appropriate $\mathscr{H}$ so that the algebra $\mathbf{M}_2 = \langle \mathbf{M}_1, \mathbf{e}_1 \rangle$ generated by $\mathbf{M}_1$ and $\mathbf{e}_1$ makes sense. He then investigates $\mathbf{M}_2$ and proves that it is another type II_1 factor, which contains $\mathbf{M}_1$ as a subfactor, moreover the index $|\mathbf{M}_2 : \mathbf{M}_1|$ is equal to the index $|\mathbf{M}_1 : \mathbf{M}_0|$, i.e. to λ. Having in hand another II_1 factor $\mathbf{M}_2$ and its subfactor $\mathbf{M}_1$, there is also a trace on $\mathbf{M}_2$ which (by the uniqueness of the trace) coincides with the trace on $\mathbf{M}_1$ when it is restricted to $\mathbf{M}_1$, and another conditional expectation $\mathbf{e}_2 : \mathbf{M}_2 \to \mathbf{M}_1$. This allows Jones to iterate the construction, to build algebras $\mathbf{M}_1, \mathbf{M}_2 \ldots$ and from them a family of algebras:

$$\mathbf{J}_n = \{\mathbf{1}, \mathbf{e}_1, \ldots, \mathbf{e}_{n-1}\} \subset \mathbf{M}_n, \quad n = 1,2,3,\ldots\ .$$

Rewriting history a little bit in order to make the subsequent connection with knots a little more transparent, we now replace the $\mathbf{e}_k$'s by a new set of generators which are units, defining:

$$\mathbf{g}_k = q\mathbf{e}_k - (1 - \mathbf{e}_k),$$

where

$$(1-q)(1-q^{-1}) = 1/\lambda.$$

The $\mathbf{g}_k$'s generate $\mathbf{J}_n$, because the $\mathbf{e}_k$'s do, and we can solve for the $\mathbf{e}_k$'s in terms of the $\mathbf{g}_k$'s. So

$$\mathbf{J}_n = \mathbf{J}_n(q) = \{\mathbf{1}, \mathbf{g}_1, \ldots, \mathbf{g}_{n-1}\},$$

and we have a *tower of algebras*, ordered by inclusion:

$$\mathbf{J}_1(q) \subset \mathbf{J}_2(q) \subset \mathbf{J}_3(q) \subset \ldots\ .$$

The parameter q, which replaces the index λ, is the quantity now under investigation.

The parameter q is woven into the construction of the tower. First, defining relations in $\mathbf{J}_n(q)$ depend upon q:

$$\text{(1)} \qquad \mathbf{g}_i\mathbf{g}_k = \mathbf{g}_k\mathbf{g}_i \quad \text{if } |i-k| \geq 2,$$

$$\text{(2)} \qquad \mathbf{g}_i\mathbf{g}_{i+1}\mathbf{g}_i = \mathbf{g}_{i+1}\mathbf{g}_i\mathbf{g}_{i+1},$$

$$(3_q) \qquad \mathbf{g}_i^2 = (q-1)\mathbf{g}_i + q,$$

$$\text{(4)} \qquad \mathbf{1} + \mathbf{g}_i + \mathbf{g}_{i+1} + \mathbf{g}_i\mathbf{g}_{i+1} + \mathbf{g}_{i+1}\mathbf{g}_i + \quad + \mathbf{g}_i\mathbf{g}_{i+1}\mathbf{g}_i = 0.$$

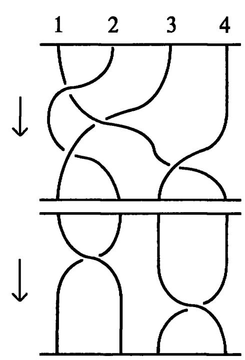
A second way in which q enters into the structure involves the trace. Recall that since $\mathbf{M}_n$ is type II_1 it supports a unique trace, and since $\mathbf{J}_n$ is a subalgebra it does too, by restriction. This trace is known as a *Markov trace*, i.e. it satisfies the important property:

$$(5_q) \qquad \mathrm{tr}(\mathbf{w}\mathbf{g}_n) = \tau(q)\,\mathrm{tr}(\mathbf{w}) \quad \text{if } \mathbf{w} \in \mathbf{J}_n,$$

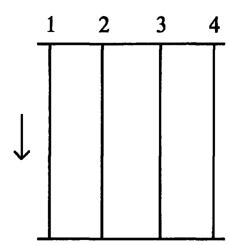
where $\tau(q)$ is a fixed function of q. Thus, for each fixed value of q the trace is multiplied by a fixed scalar when one passes from one stage of the tower to the next, by multiplying an arbitrary element of $\mathbf{J}_n$ by the new generator $\mathbf{g}_n$ of $\mathbf{J}_{n+1}$.

Relations (1) and (2) above have an interesting geometric meaning, familiar to topologists. They are defining relations for the *n-string braid group*, $\mathbf{B}_n$, discovered by Emil Artin [Ar] in a foundational paper written in 1923. We pause to discuss braids.

An n-braid may be visualized by a weaving pattern of strings in 3-space which join n points on a horizontal plane to n corresponding points on a parallel plane, as illustrated in the example in Figure 2, where $n = 4$. In the case $n = 3$, the familiar braid in a person's hair gives another example. The strings are allowed to be stretched and deformed, the key features being that strings cannot pass through one-another and always proceed directly downward in their travels from the upper plane to the lower one. The equivalence class of weaving patterns under such deformations is an *n-braid*. One multiplies braids by concatenation and erasure of the middle plane. This multiplication makes them into a group, the n-string braid group $\mathbf{B}_n$. The identity is a braid which, when pulled taut, goes over to n straight lines. Generators are the $n-1$ elementary braids which (by an abuse of notation) we continue to call $\mathbf{g}_1, \ldots, \mathbf{g}_{n-1}$. The pictures in Figure 3 show that relations (1) and (2) hold between the generators of $\mathbf{B}_n$. In fact, Artin proved they are *defining relations* for $\mathbf{B}_n$. Thus for each n there is a homomorphism from the n-string braid group $\mathbf{B}_n$ into the Jones algebra $\mathbf{J}_n(q)$, and from the group algebra $\mathbb{C}\mathbf{B}_n$ onto $\mathbf{J}_n(q)$.



Examples of 4-braids



Trivial 4-braid

Fig. 2

Returning to the business at hand, i.e. the proof of the Index Theorem, Jones next shows that properties (1), (2), (3_q) and (5_q) suffice for the calculation of the

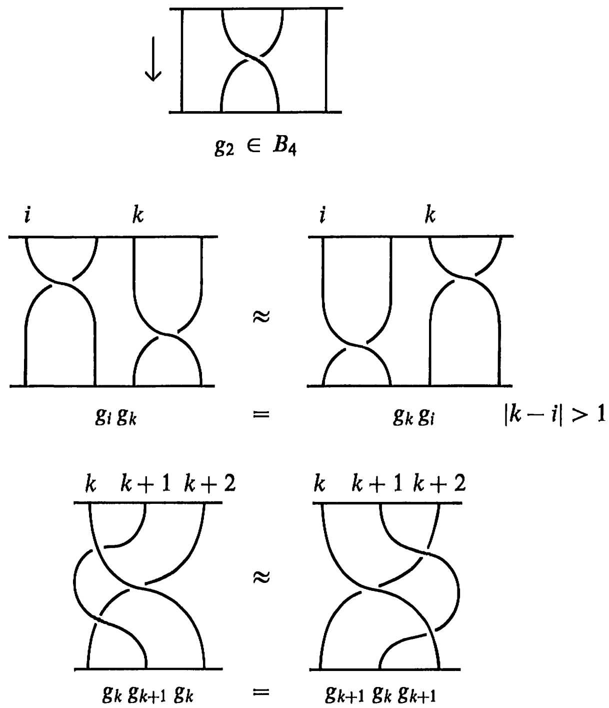


Fig. 3. Generators and defining relations in $\mathbf{B}_n$

trace of an arbitrary element $\mathbf{x} \in \mathbf{J}_n(q)$. It turns out that trace($\mathbf{x}$) is an *integer polynomial* in $(\sqrt{q})^{\pm 1}$. (We will meet it again in a few moments as the Jones polynomial associated to $\mathbf{x}$.) Jones proof of the Index Theorem is concluded when he shows that the infinite sequence of algebras $\mathbf{J}_n(q)$, with the given trace, could not exist if q did not satisfy the restrictions of the Index Theorem.

2. Knots and Links

We have already seen hints of topological meaning in $\mathbf{J}_n(q)$ via braids. There is more to come. Knots and links are obtained from braids by identifying the initial points and end points of a braid in a circle, as illustrated in Figure 4. It was proved by J. W. Alexander in 1928 that every knot or link arises in this way. Earlier we described an equivalence relation on weaving patterns which yields braids, and there is a similar (but less restrictive) equivalence relation on knots, i.e. a knot or link *type* is its equivalence class under isotopy in 3-space. Note that isotopy in 3-space which takes one closed braid representative of a link to another closed braid representative will pass through a sequence of representatives which are not closed braids in an obvious way. For example see the 2-component link which is illustrated in Figure 4. The left picture is an obvious closed braid representative, whereas the right is not.

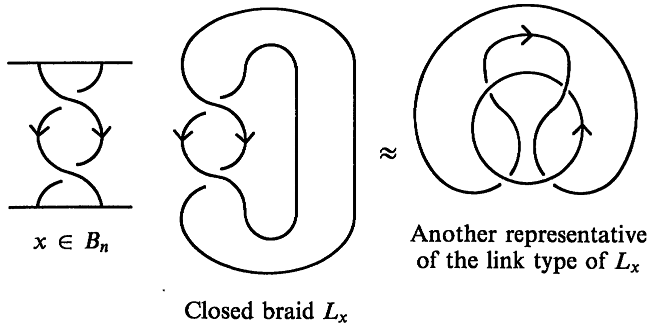


Fig. 4. Braids determine links

Let $\mathbf{B}_\infty$ denote the disjoint union of all of the braid groups $\mathbf{B}_n$, $n = 1, 2, 3, \ldots$. In 1935 the mathematician A. A. Markov proposed the equivalence relation on $\mathbf{B}_\infty$ which corresponds to link equivalence [M]. Remarkably, the properties of the trace, or more particularly the facts that $\mathrm{tr}(\mathbf{xy}) = \mathrm{tr}(\mathbf{yx})$ together with property (5_q), were exactly what was needed to make the trace polynomial into an invariant on Markov's equivalence classes! Using Markov's proposed equivalence relation (which was proved to be the correct one in 1972 [Bi]), Jones proved, with almost no additional work beyond results already established in [Jo1], the following theorem:

Theorem [Jo3]. *If* $\mathbf{w} \in \mathbf{B}_\infty$, *then (after multiplication by an appropriate scalar, which depends upon the braid index n) the trace of the image of* $\mathbf{w}$ *in* $\mathbf{J}_n(q)$ *is a polynomial in* $(\sqrt{q})^{\pm 1}$ *which is an invariant of the link type defined by the closed braid* $\mathbf{L}_\mathbf{w}$.

The invariant of Jones' theorem is the one-variable *Jones polynomial* $\mathbf{V}_x(q)$. Notice that the independent "variable" in this polynomial is essentially the index of a type II_1 subfactor in a type II_1 factor! It's discovery opened a new chapter in knot and link theory.

3. Statistical Mechanics

We promised to discuss other ways in which the work of Jones was related to yet other areas of mathematics and physics, and begin to do so now. As it turned out, when Jones did his work the family of algebras $\mathbf{J}_n(q)$ were already known to physicists who were concerned with *exactly solvable models* in Statistical Mechanics. (For an excellent introduction to this topic, see R. Baxter's article in these Proceedings.) One of the simplest examples in this area is known as the *Potts model.* In that model one considers an array of "atoms" arranged at the vertices of a planar lattice with m rows and n columns as in Figure 5. Each "atom" in the system has various possible spins associated to it, and in the simplest case, known as the Ising model, there are two choices, "+" for spin up and "−" for spin down. We have indicated one of the 2^{nm} choices in Figure 5, determining a

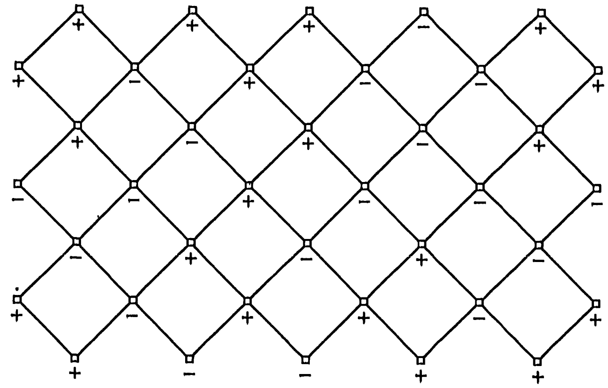


Fig. 5

state of the system. The goal is to compute the free energy of the system, averaged over all possible states.

Letting σ_i denote the spin at site i, we note that an edge e contributes an *energy* $E_e(\sigma_i, \sigma_j)$, where σ_i and σ_j are the states of the endpoints of e. Let E be the collection of lattice edges. Let β be a parameter which depends upon the temperature. Then the Gibbs *partition function* $\mathbf{Z}$ is given by the formula:

$$\mathbf{Z} = \sum_{\sigma_1,\ldots,\sigma_{mn}} \prod_{e \in E} \exp(-\beta E_e(\sigma_i, \sigma_j)).$$

All of this is microscopic, nevertheless the major macroscopic thermodynamic quantities are functions of the partition function. In particular, the free energy $\mathbf{F}$, the object of interest to us at this time, is given by $\mathbf{Z} = \exp(-\beta\mathbf{F})$.

To compute the manner in which the atoms in one row of the lattice interact with atoms in the next, physicists set up the *transfer matrix* $\mathbf{T}$, which expresses the row-to-row interactions. It turned out that, in order to be able to calculate the free energy, the transfer matrices must satisfy conditions known as the *Yang-Baxter equations* and (to the great surprise of everyone) they turned out to be the braid relations (1) and (2) in disguise! (Remark: before Jones' work, to the best of our knowledge, it was not known that the Yang-Baxter equation was related to braids or knots.) Even more, the algebra which the transfer matrices generate in the Ising model, known to physicists as the *Temperley-Lieb algebra*, is our algebra $\mathbf{J}_n(q)$. The partition function $\mathbf{Z}$ is related in a very simple way to the transfer matrix:

$$\mathbf{Z} = \text{trace}(\mathbf{T})^m.$$

In fact, it is closely related to the Jones trace.

The initial discovery of a relationship between the Potts model and links was reported on in [Jo3]. It opened a new chapter in the flow of ideas between mathematics and physics. We give an explicit example of a way in which the relationship of Jones' work to physics led to new insight into mathematics. Learning that the partition function was a sum over states of the system, Louis Kauffman was led to seek a decomposition of the Jones polynomial into a related sum over "states" of knot diagram, and arrived in [K1] at an elegant "states model" for the Jones polynomial. The Jones polynomial, and Kauffman's states model for it, were later seen to generalize to other polynomial invariants, with associated states models, for links in S^3 and eventually into invariants for 3-manifolds M^3 and links in 3-manifolds. The full story is not known at this writing, however we refer the reader to V. Turaev's article in these *Proceedings* for an excellent account of it, as of August 1990.

4. Quantum Groups and Representations of Lie Algebras

We begin by explaining the structure of the algebra $\mathbf{J}_n(q)$. It will be convenient to begin with another algebra $\mathbf{H}_n(q)$, which is generated by symbols $\mathbf{g}_1, \ldots, \mathbf{g}_{n-1}$ (which now have a third meaning), with defining relations (1), (2) and (3_q). The algebra $\mathbf{H}_n(q)$ is very well-known to mathematicians. It's the *Iwahori-Hecke algebra*, also known as the *Hecke algebra of the symmetric group* [Bo]. Its relationship with the symmetric group is simple to describe and beautiful. Notice that when $q = 1$, relation (3_q) simplifies to $(\mathbf{g}_k)^2 = 1$. One recognizes (1), (2) and (3_1) as defining relations for the group algebra $\mathbb{C}\mathbf{S}_n$ of the symmetric group $\mathbf{S}_n$. Here $\mathbf{g}_k$ is to be re-interpreted as a transposition which exchanges the symbols $\mathbf{k}$ and $\mathbf{k+1}$. In this way we may view $\mathbf{H}_n(q)$ as a "q-deformation" of the complex group algebra $\mathbb{C}\mathbf{S}_n = \mathbf{H}_n(1)$.

The algebra $\mathbb{C}\mathbf{S}_n$ is *rigid*, that is if one deforms it in this way its irreducible summands continue to be irreducible summands of the same dimension, in fact $\mathbf{H}_n(q)$ is actually algebra-isomorphic to $\mathbb{C}\mathbf{S}_n$ for generic q. Thus $\mathbf{H}_n(q)$ is a direct sum of finite dimensional matrix algebras, its irreducible summands being in one-to-one correspondence with the irreducible representations of the symmetric group $\mathbf{S}_n$. In this setting, Jones showed in [Jo2] that for generic q the algebra $\mathbf{J}_n(q)$ may be interpreted as the algebra associtated to the q-deformations of those irreducible representations of $\mathbf{S}_n$ which have Young diagrams with at most two rows.

We now explain how $\mathbf{H}_n(q)$ is related to quantum groups. It will be helpful to recall the classical picture. The fundamental representation of the Lie group GL_n acts on $\mathbb{C}^n$, and so its k-fold tensor product acts naturally on $(\mathbb{C}^n)^{\otimes k}$. The symmetric group $\mathbf{S}_k$ also acts naturally on $(\mathbb{C}^n)^{\otimes k}$, permuting factors. (Remark: In this latter action, the representations of $\mathbf{S}_k$ which are relevant are those whose Young diagrams have $\leq n$ rows.) As is well known, the actions of GL_n and $\mathbf{S}_k$ are each other's commutants in the full group of linear transformations of $(\mathbb{C}^n)^{\otimes k}$. If one now replaces GL_n and $\mathbb{C}\mathbf{S}_k$ by the quantum group $U_q(GL_n)$ and the Hecke algebra $\mathbf{H}_k(q)$ respectively, then the remarkable fact is that $U_q(GL_n)$ and $\mathbf{H}_k(q)$ are still each other's commutants [Ji]. The corresponding picture for $\mathbf{J}_n(q)$ is obtained by restricting to GL_2 and to representations of $\mathbf{S}_k$ having Young diagrams with at most 2 rows.

We remark that these are not isolated instances of algebraic accidents, but rather special cases of a phenomenon which relates a large part of the mathematics of quantum groups to finite dimensional matrix representations of the group algebra $\mathbb{C}\mathbf{B}_n$ which support a Markov trace (e.g. see [BW]).

5. Dynkin Diagrams

Dynkin diagrams arise in the tower construction which we described in § 1 via the inclusions of the algebras $\mathbf{J}_n(q)$ in the Jones tower. The inclusions for the Jones tower are very simple, and correspond to the Dynkin diagram of type A_n. However, other, more complicated towers may be obtained by replacing the II_1 factor $\mathbf{M}_1$ in the tower construction of § 1 above by $\mathbf{M}_1 \cap (\mathbf{M}_0)'$, where $(\mathbf{M}_0)'$ is the commutant of $\mathbf{M}_0$ in $\mathbf{M}_1$. We refer the reader to [GHJ] for an introduction to this topic and a discussion of the "derived tower" and the Dynkin diagrams which occur. The connections with the representations of simple Lie algebras can be guessed at from our discussion in § 4 above.

6. Concluding Remarks

I hope I have succeeded in showing you some of the ways in which Jones' work created bridges between the areas of mathematics which were illustrated in Figure 1. To conclude, I want to indicate very briefly some of the ways in which those bridges have changed the mathematics which many of us are doing.

There is another link polynomial in the picture, the famous *Alexander polynomial*. It was discovered in 1928, and was of fundamental importance to knot theory, both in the classical case of knots in S^3 and in higher dimensional knotting. Shortly after Jones' 1984 discovery, it was learned that in fact both the Alexander and Jones polynomial were specializations of the 2-variable Jones polynomial. That discovery was made simultaneously by five separate groups of authors: Freyd and Yetter, Hoste, Lickorish and Millett, Ocneanu, and Przytycki and Traczyk, a simple version of the proof of Ocneanu being given in [Jo3]. One of the techniques used in the proof was the combinatorics of link diagrams, and that technique led to the discovery of yet another polynomial, by Louis Kauffman [K2].

From the point of view of algebra, the Jones polynomial comes from a trace function on $\mathbf{J}_n(q)$, and the 2-variable Jones polynomial from a similar trace on the full Hecke algebra $\mathbf{H}_n(q)$. Beyond that, there is another algebra, the so-called Birman-Wenzl algebra [BW], and Kauffman's polynomial is a trace on it. Even more, physicists who had worked with solutions to the Yang Baxter equation, realized that they knew of still other Markov traces, so they began to grind out still other polynomials, in initially bewildering confusion. That picture is fairly well understood at this moment, however the work of Witten [W] indicates there are still other, related, link invariants. The generalizations are vast, with much work to be done.

There is also a different and very direct way in which Jones has had equal influence. His style of working is informal, and one which encourages the free and open interchange of ideas. During the past few years Jones wrote letters to various people which described his important new discoveries at an early stage, when he did not yet feel ready to submit them for journal publication because he had much more work to do. He nevertheless asked that his letters be shared, and so they were widely circulated. It was not surprising that they then served as a rich source of ideas for the work of others. As it has turned out, there has been more than enough credit to go around. His openness and generosity in this regard have been in the best tradition and spirit of mathematics.

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