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HISTORICAL ARTICLE

Life and Work of Friedrich Hirzebruch

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Abstract Friedrich Hirzebruch, who died in 2012 at the age of 84, was one of the most important German mathematicians of the twentieth century. In this article we try to give a fairly detailed picture of his life and of his many mathematical achievements, as well as of his role in reshaping German mathematics after the Second World War.

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Friedrich Hirzebruch, who passed away on May 27, 2012, at the age of 84, was the outstanding German mathematician of the second half of the twentieth century, not only because of his beautiful and influential discoveries within mathematics itself, but also, and perhaps even more importantly, for his role in reshaping German mathematics and restoring the country's image after the devastations of the Nazi years. The field of his scientific work can best be summed up as "Topological methods in algebraic geometry," this being both the title of his now classic book and the aptest description of an activity that ranged from the signature and Hirzebruch-Riemann-Roch theorems to the creation of the modern theory of Hilbert modular varieties. Highlights of his activity as a leader and shaper of mathematics inside and outside Germany include his creation of the Arbeitstagung, his presidency of the Deutsche Mathematiker-Vereinigung during two especially critical periods and his later services to the European Mathematical Society and the International Mathematical Union, the founding of the Max Planck Institute for Mathematics in Bonn, and his role in preserving mathematical contacts between the Federal Republic of Germany and the German Democratic Republic and Soviet Union during the Soviet period and later in establishing close mathematical links between Germany and many other countries, notably Japan,

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Fig. 1 F. Hirzebruch with the author at the Max Planck Institute for Mathematics ca. 1985

Poland and Israel. He was a superb lecturer, teacher, and expositor of mathematics and above all a man whose human qualities were an inspiration and a model for those around him.

Accounts of several periods of Hirzebruch's life and activities, told in his inimitable style, can be found in a number of his own articles, while more systematic accounts are given in the article by Joel Segel, the long video interview with Matthias Kreck for the archives of the Simon Foundation, and the AMS memorial article edited by Michael Atiyah and myself, which also contains portraits of him by several of his friends and associates. The exact references for all of these are given in the short bibliography at the end of this article. A book-length biography by Winfried Scharlau is also in preparation. The emphasis here is different: I would like to both recount the main stages of Hirzebruch's career and to give a somewhat more in-depth look at some of his mathematics and at the many ways in which he helped shape the development of mathematics during his time. The article will be organized roughly chronologically, but alternating between the events and activities of his life and a description of his mathematics.

This article owes much to Michael Atiyah and Matthias Kreck, with whom I had many helpful discussions. Above all I would like to thank my wife, Silke Wimmer-Zagier, who participated in every step of the writing. Our friendship with Fritz Hirzebruch transformed both of our lives.

1 Early Years and Education

Friedrich ("Fritz") Ernst Peter Hirzebruch was born to Dr. Fritz Hirzebruch and Martha Hirzebruch, née Holtschmit, on October 17, 1927, in Hamm, North Rhine-Westphalia. His love of mathematics was awakened early by his father, a mathematics teacher and headmaster of the local secondary school, one of whose students, Karl Stein, was later to become one of Hirzebruch's mentors and a famous mathematician in his own right. Already as a child he was fascinated by concepts like other number systems or the irrationality of $\sqrt{2}$, and apart from a brief flirtation with astronomy he never seriously imagined pursuing any other profession than that of a mathematician.

His youth was marked by the Third Reich and the Second World War. While still a teenager he was drafted into "Deutsches Jungvolk" (German Youth) and assigned to anti-aircraft positions, where he officially scanned the skies for enemy planes but mostly thought about mathematics. He was drafted into the army soon before Germany's collapse, escaping active combat by a stroke of luck, and soon afterwards was captured by Allied forces and put into a POW camp, where he continued to do mathematics, famously writing some of it on toilet paper. At that time, whenever asked about his war experiences, he would answer laconically with the single sentence "In the period from mid-January 1945 to the first of July 1945 I went through fatigue duty, military service, and war captivity." He was released in July 1945.

Hirzebruch began his undergraduate studies in December 1945, soon after his 18th birthday and, of course, also soon after the end of the war. The university of Münster was in ruins and in the first months the unique auditorium available to the Faculty of Philosophy and Science could only be used for mathematics lectures one day every three weeks, but the situation gradually got better, with lectures every two weeks in the following spring and every week by Fall 1946.

His teachers at Münster were Heinrich Behnke and the above-mentioned Karl Stein. From them he learned the function theory of several complex variables that was to be his first field of research and the subject of his thesis. During his sixth semester, in 1948, he saw an advertisement for a three-week stint of agricultural work in Switzerland, to be followed by a fourth week in a freely chosen place in Switzerland, and decided to apply. With the support of Behnke and Heinrich Scholz, for whose seminar on logic and foundations he had been one of the student assistants, he was awarded this "prize," which involved 14 hours of hard physical labor a day for the first three weeks, with a financial remuneration of 40 Swiss francs during the last week. His stay in Switzerland was to determine the course of his future mathematical life.

2 Switzerland, 1948–1950

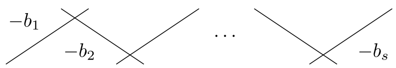
Before Hirzebruch left Münster, Behnke had written to Heinz Hopf in Zürich to ask whether he would receive and advise the young Fritz. The recommendation letter he sent was also memorable because the secretary to whom it was dictated, Ingeborg Spitzley, soon afterwards became a mathematics student and four years later Hirzebruch's wife!

After completing his stint as a farm laborer, which taxed his weakened constitution to the utmost but which he seems to have rather enjoyed nevertheless, he went to Zürich to visit Hopf and Paul Bernays. He was received with extraordinary friendliness (this was only three years after the end of the war) by the Hopfs, who put him up in their own house and treated him like a son. From Hopf he had his first introduction into differential topology, with Hopf's brand-new article "Zur Topologie der komplexen Mannigfaltigkeiten" as the guiding text. Here he learned about almost complex structures and how to show that such manifolds as S^4, S^8 or the complex projective plane with reversed orientation do not have any and hence in particular cannot be complex manifolds. In [12] he writes that these first discussions were the starting point for both his own first publication, in Crelle's Journal in 1953, and his later joint paper with Hopf, in the Mathematische Annalen in 1958.

After his return to Münster, Hirzebruch continued his studies with Behnke, Stein and Scholz for one more semester, until Spring 1949. With the support of his teachers, a scholarship for him to study at the ETH in Zürich was set up, with a matching fund established that the ETH could use later for a Swiss student to study in Germany (as indeed happened, in 1951). He spent the period from April 1949 till August 1950 at the ETH, receiving their Silver Medal (and a prize of 1000 Swiss francs) for his solution of the prize problem of developing the relation between differential topology and the theory of analytic functions in many variables ("now, who might have suggested *that* topic?" he asks drily in [12]) in July 1950 and completing his doctoral dissertation at the same time. He had to return to Münster to defend it, in the same month, because his official supervisor was still Heinrich Behnke, but he himself always said that his true doctoral supervisor was Heinz Hopf, who sometimes referred to him as his illegitimate *Doktorsohn*.

The subject of the thesis was the resolution of singularities (or "Hopf σ-process") on complex algebraic surfaces. Here he found a toroidal construction (as such things later came to be called) and discovered that by iteration of the σ- or blowing-up process one often obtained a graph of rational curves whose self-intersections were given by the partial quotients of a continued fraction, a harbinger of the links to number theory that were to be one of the main characteristics of his later work. In particular, he showed that the resolution of the singularity $w^n = z_1 z_2^{n-q}$, where $0 < q < n$ are coprime integers, is given by a chain of s rational curves with any two successive curves intersecting in one point and with self-intersections $-b_1, \dots, -b_s$, where s and the b_i (≥ 2) are given by the "minus" continued fraction of n/q:

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_s}}}$$



In this connection he made the incorrect assertion that the resolution process could obviously never produce a cycle—an error that he noticed soon afterwards and for which he would atone most handsomely many years later (as we will discuss in detail below) with his discovery of the resolution of the cusp singularities of Hilbert modular surfaces by cycles of rational curves related to periodic continued fractions.

In this thesis Hirzebruch introduced the family of algebraic surfaces Σ_n that are now called *Hirzebruch surfaces* and proved that the manifolds Σ_n and Σ_m are homeomorphic whenever $n - m$ is even, but are always distinct as algebraic or complex surfaces. From the modern point of view this is a simple result: The surfaces Σ_n are the total spaces of certain bundles, so the classification of bundles together with the intersection form on cohomology gives the homeomorphism classification, and the first Chern class distinguishes them in the complex world. But at this time these fundamental concepts were either still unknown or had just been discovered, and Hirzebruch himself was to become one of the main people to contribute to their study.

Hirzebruch returned to Germany in the winter semester of 1950 to take up a position as *Assistent* in Erlangen, where he stayed for two years and published the results of his thesis and other related work. But it was in Princeton, where he spent the two following years, that his mathematics really took off.

3 The Golden Years: Princeton 1952–54

Hirzebruch himself always used the words “golden years” when speaking about the time he spent at the Institute for Advanced Study in Princeton. It was a period of intense excitement for him, both mathematically and personally. He had received an invitation, probably engineered by Heinz Hopf, in Erlangen, and left by boat on August 9, 1952; his bride Inge (Ingeborg, née Spitzley), whom he had married in a civil ceremony just before leaving, joined him in November in Princeton, where they celebrated a church wedding. Hirzebruch’s name was already known to some of his American colleagues before he arrived, and he was met in person by Newton Hawley and Donald Spencer when his ship docked in Hoboken. He began working with them and Kunihiko Kodaira right away, in particular meeting Spencer and Kodaira (who were themselves in the middle of their now-famous collaboration) every day from 9 till 5 at the Institute [6]. He thus got to know “from the horse’s mouth” the newest and most up-to-date advances in algebraic geometry, the theory of sheaves, bundles, and characteristic classes, whose definitive formulation and exposition he was to give in the book *Neue topologische Methoden in der algebraischen Geometrie*.

The institute was full of outstanding personalities, not only the famous scientists who had come from Germany and Eastern Europe before the war like Albert Einstein (whom Inge once famously mistook for the gardener), Kurt Gödel (whom Hirzebruch remembered always seeing immersed in conversation with Einstein), Marston Morse (whose seminars Hirzebruch attended), or Hermann Weyl (who took great interest in Hirzebruch’s work and befriended both him and his wife), but also a whole generation of brilliant young European mathematicians who were to shape the coming decades and many of whom became Hirzebruch’s close friends: Borel, Bott, Atiyah, Lang, Singer, Serre, Milnor, and many others. The mathematics that they discussed and developed during this period were to shape all of Hirzebruch’s future work, as well as much of the mathematics of the second half of the twentieth century, and in the case of two of them, Atiyah and Borel, led to extremely fruitful collaborations that we will discuss below in detail.

A high point of Hirzebruch’s two years in Princeton was the birth of his first child, Ulrike, in 1953. (His other two children were born after his return to Germany, Barbara in Hamm in 1956 and Michael in Bonn in 1958. Ulrike and Barbara both later studied mathematics and worked in related fields, Ulrike in mathematics publishing and Barbara as a teacher, while Michael became a doctor.) But these years also marked the birth of the two great theorems that bear his name, the Hirzebruch Signature Theorem and the Hirzebruch-Riemann-Roch Theorem. We now describe each of these in turn.

4 Characteristic Classes, Genera, and the Signature Theorem

In topology, as in the rest of mathematics, one wants to find computable invariants that permit one to distinguish non-isomorphic objects, where “non-isomorphic” here might mean “non-homeomorphic” or “non-diffeomorphic” according to the situation. These invariants can be purely numeric, or else algebraic objects like the homology or cohomology of a manifold (each of which is a graded vector space endowed with additional structure, the intersection form or cup product, respectively), or yet more subtle invariants like characteristic classes in the cohomology ring. The most interesting invariants often have more than one description. For instance, the Euler characteristic $e(M)$ of a closed manifold M can be defined purely combinatorially as the alternating sum of the numbers of simplices of various dimensions in a triangulation of M (but then one has to demonstrate that this definition is independent of the triangulation, as was done famously by Euler for the 2-sphere, with Euler number 2), or algebraically as the alternating sum of the dimensions of its homology or cohomology groups, or, in the case when M is a complex n-manifold, as the integral of the nth (or total) Chern class of the tangent bundle of M:

$$e(M) = \int_M c(TM) = \langle c_n(TM), [M] \rangle .$$

The signature theorem gives a similar description of the signature, which is an important topological invariant of oriented manifolds. (We recall briefly the definition: if M is a closed oriented manifold of even dimension n, then the intersection form on $H^{n/2}(M)$ is non-degenerate by Poincaré duality and is symmetric if $n/2$ is even, in which case $\mathrm{Sign}(M)$ is defined to be the signature of this form, namely, the number $r - s$ if the form can be written as $x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots x_{r+s}^2$ for some basis $x_1, \ldots, x_{r+s}$ of $H^{n/2}(M; \mathbb{R})$. If $4 \nmid \dim M$, then $\mathrm{Sign}(M) = 0$.) To explain it, we must first say something about characteristic classes, to whose theory Hirzebruch contributed substantially (in particular through his work with A. Borel, discussed below), and also about multiplicative sequences and genera, which are among the most wonderful of all of his inventions, beautifully combining theoretical and computational ideas.

We start with characteristic classes, the two main cases for our discussion being the Chern classes $c_i(E) \in H^{2i}(X)$ of a complex vector bundle E over a topological space X and the Pontryagin classes $p_i(E) \in H^{4i}(X)$ of a real vector bundle E over X. (A third case, the Stiefel-Whitney classes in $H^*(X; \mathbb{Z}/2\mathbb{Z})$ of a real vector bundle over X, will be omitted, although it plays a role in several of Hirzebruch’s works.) The basic one is the Chern class. Recall that a (topological) d-dimensional complex vector bundle E over X is a map $p : E \to X$ that “looks locally like a product with $\mathbb{C}^d$”, in the sense that X can be covered by small open sets U with isomorphisms between $p^{-1}(U)$ and $U \times \mathbb{C}^d$ that are fixed up to the action of a continuous map from U to $\mathrm{GL}(d, \mathbb{C})$. In the case $d = 1$ (complex line bundles), the bundle E is uniquely determined by a characteristic class $c_1(E) \in H^2(X; \mathbb{Z})$, called its first Chern class, and we define the total Chern class of E to be $c(E) = 1 + c_1(E)$. We also require the total Chern class of a direct sum of two bundles to be the product of their total Chern classes, so that if a d-dimensional bundle E is a sum of d line bundles $L_1, \ldots, L_d$, with first Chern classes $x_1, \ldots, x_d$, then we have

$$c(E) = \prod_{\alpha=1}^{d} (1 + x_\alpha) = \sum_{i=0}^{d} c_i(E),$$

with $c_i(E) \in H^{2i}(X;\mathbb{Z})$ being the ith elementary symmetric polynomial in the x_j. It turns out that these requirements and the functoriality property $c(f^*(E)) = f^*(c(E))$ for the pull-back of a bundle E over X by a continuous map $f : Y \to X$ already suffice to define Chern classes in general (by the so-called splitting principle). Finally, Pontryagin classes can be defined in terms of Chern classes: if E is a real vector bundle over X (defined like complex bundles, but with $\mathbb{C}$ replaced by $\mathbb{R}$), then $E \otimes_{\mathbb{R}} \mathbb{C}$ is a complex bundle having (up to 2-torsion) non-zero Chern classes only for even indices, and one defines the individual and total Pontryagin classes of E by $p_i(E) = (-1)^i c_{2i}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(X)$ and $p(E) = \sum_i p_i(E) \in H^{4*}(X)$, respectively. We should emphasize that the algebraic approach described here was developed to a large extent by Hirzebruch and his collaborators (especially Atiyah and Borel), and is more general and easier to work with than the approach based on geometry as used earlier by Pontryagin, Chern and others.

We now come to Hirzebruch's multiplicative sequences. These are multiplicative characteristic classes (i.e., functions assigning to any bundle E over a space X an element $\chi(E) \in H^*(X;R)$ and satisfying $\chi(f^*E) = f^*(E)$ and $\chi(E_1 \oplus E_2) = \chi(E_1)\chi(E_2)$) that are canonically associated to the Chern or Pontryagin class of E and to a power series $f(x) = 1 + a_1x + a_2x^2 + \cdots$ with coefficients in some ring R (for us usually $\mathbb{Q}$). In the complex case the multiplicative sequence $\underline{\chi}_f$ is defined by formally factorizing the Chern class $c(E)$ as $\prod_\alpha(1 + x_\alpha)$ with each x_α of degree 2 and setting $\underline{\chi}_f(E) = \prod_\alpha f(x_\alpha)$, which makes sense because the product

$$\begin{aligned}\prod_\alpha f(x_\alpha) &= \prod_\alpha\bigl(1 + a_1x_\alpha + a_2x_\alpha^2 + \cdots\bigr)\\ &= 1 + a_1\sum_\alpha x_\alpha + \left(a_2\sum_\alpha x_\alpha^2 + a_1^2\sum_{\alpha<\beta} x_\alpha x_\beta\right) + \cdots\\ &= 1 + a_1c_1(E) + \bigl(a_2c_1(E)^2 + \bigl(a_1^2 - 2a_2\bigr)c_2(E)\bigr) + \cdots\end{aligned}$$

can be written completely in terms of the elementary symmetric functions $c_i(E)$ of the formal roots x_α. (A trivial case is obtained by taking $f(x) = 1 + x$, in which case $\underline{\chi}_f(E)$ is simply $c(E)$.) An exactly similar construction works for real bundles and Pontryagin classes, where we write $p(E)$ as $\prod_\alpha(1 + y_\alpha)$ with formal roots y_α of degree 4 and define $\underline{\underline{\chi}}_h(E) \in H^{4*}(X)$ for any power series $h(y)$ by $\underline{\underline{\chi}}_h(E) = \prod_\alpha h(y_\alpha)$, or equivalently by $\underline{\underline{\chi}}_h(E) = \underline{\chi}_f(E \otimes_{\mathbb{R}} \mathbb{C})$ where $f(x)$ is the even power series $h(-x^2)$. In particular, if

$$h(y) = \frac{\sqrt{y}}{\tanh\sqrt{y}} = 1 + \frac{y}{3} - \frac{y^2}{45} + \frac{2y^3}{945} + \cdots$$

(corresponding to $f(x) = x/\tan x$), then we get the total *Hirzebruch L-class*

$$\mathcal{L}(E) = \underline{\underline{\chi}}_{\sqrt{y}/\tanh\sqrt{y}} = 1 + \mathcal{L}_1(E) + \mathcal{L}_2(E) + \cdots$$

of the real bundle E, where each $\mathcal{L}_i(E)$ belongs to $H^{4i}(X;\mathbb{Q})$, the first values being

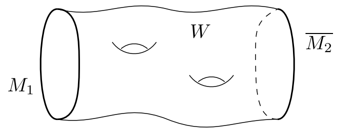
$$\mathcal{L}_1 = \frac{p_1}{3}, \qquad \mathcal{L}_2 = \frac{-p_1^2 + 7p_2}{45}, \qquad \mathcal{L}_3 = \frac{2p_1^3 - 13p_1p_2 + 62p_3}{945}, \qquad \ldots.$$

Hirzebruch's Signature Theorem states that, for all oriented manifolds M, one has

$$\mathrm{Sign}(M) = \big\langle \mathcal{L}(TM), [M] \big\rangle,$$

where TM is the tangent bundle of M. About its proof, he liked to say: "Having conjectured the signature theorem, I went to the library at Fine Hall. Thom's *Comptes Rendus* note on cobordism had just appeared. This finished the proof." This modest statement is nearly true, but only because of a key insight that Hirzebruch had already had, namely, that every multiplicative sequence $\underline{\underline{\chi}}_h$ defines a "genus." To explain this, we first recall the concept of cobordism.

Two closed oriented n-manifolds M_1 and M_2 are called *cobordant* if the disjoint union of M_1 and $\overline{M}_2$ ($= M_2$ with reversed orientation) bounds an oriented $(n+1)$-manifold.



The equivalence classes under this equivalence relation form an abelian group Ω_n, with addition given by disjoint union and negation by change of orientation (which works because $M \sqcup \overline{M}$ bounds $M \times [0,1]$), and $\Omega_* := \bigoplus \Omega_n$ forms a graded ring via $[M_1][M_2] = [M_1 \times M_2]$ (Cartesian product). It is relatively easy to show that the signature defines a homomorphism from this ring to $\mathbb{Z}$, i.e., that it behaves additively and multiplicatively under disjoint union and Cartesian product and that it vanishes for orientable boundaries. Such a map is called a *genus*. On the other hand, if h is any power series with rational coefficients and $\underline{\underline{\chi}}_h$ the corresponding multiplicative sequence, then the function $\chi_h(M) := \langle \underline{\underline{\chi}}_h(M), [M] \rangle$ is also a genus, the additivity and multiplicativity being easy and the cobordism invariance a consequence of the fact that the boundary of an oriented manifold is null-homologous on this manifold (or simply of Stokes's theorem if one thinks of cohomology classes as represented by differential forms). This realization had led Hirzebruch to conjecture the signature theorem, and Thom's result saying that $\Omega_* \otimes \mathbb{Q}$ is the free polynomial algebra on the complex projective spaces $\mathbb{CP}_{2n}$ ($n = 1, 2, \ldots$) indeed quickly finished the proof, because it reduced the statement of the theorem for arbitrary orientable manifolds to the statement for these projective spaces, where it is easy: the cohomology of $\mathbb{CP}_{2n}$ is the truncated polynomial algebra $\mathbb{Z}[x]/(x^{2n+1})$ with $x \in H^2$ and $\langle x^{2n}, [\mathbb{CP}_{2n}] \rangle = 1$, its signature is always 1 (because $H^{2n} = \mathbb{Z} \cdot x^n$ and the self-intersection of x^n is 1), and the Chern class of its tangent bundle is $(1+x)^{2n+1}$, so

$$\begin{aligned}\big\langle \mathcal{L}\big(T(\mathbb{CP}_{2n})\big), [\mathbb{CP}_{2n}] \big\rangle &= \left\langle \left(\frac{x}{\tanh x}\right)^{2n+1}, [\mathbb{CP}_{2n}] \right\rangle \\ &= \text{Coefficient of } x^{2n} \text{ in } \left(\frac{x}{\tanh x}\right)^{2n+1} \\ &= \mathrm{Res}_{x=0}\left(\frac{dx}{(\tanh x)^{2n+1}}\right)\end{aligned}$$

$$= \mathrm{Res}_{t=0}\left(\frac{d\,\mathrm{arctanh}\,t}{t^{2n+1}}\right)$$
$$= \text{Coefficient of } t^{2n} \text{ in } \mathrm{arctanh}'(t)$$
$$= 1.$$

An exactly similar calculation leads to the following more general result.

Theorem *Every genus $\chi : \Omega_* \to \mathbb{Q}$ is equal to χ_h for a unique power series $h(y) \in 1 + y\mathbb{Q}[[y]]$, given by $h(y) = \frac{\sqrt{y}}{g^{-1}(\sqrt{y})}$ where $g(t) = \sum_{n=0}^{\infty} \chi(\mathbb{CP}_{2n}) \frac{t^{2n+1}}{2n+1}$.*

The signature theorem immediately had many applications. Rokhlin and Schwarz, and independently Thom, used it in the 50's to prove that Pontryagin numbers of manifolds, i.e., the evaluations on their fundamental classes of arbitrary polynomials in the Pontryagin classes of their tangent bundles, are combinatorial invariants; it was shown later by Novikov that they are in fact topological invariants. Even more spectacular was the theorem of Milnor in 1956 that the topological 7-sphere has more than one differentiable structure. (The possibility of the existence of exotic differentiable structures had already been pointed out by Hirzebruch in 1953.) In fact there are precisely 28 such structures, as Kervaire and Milnor showed later, where the factor 7 in the number "28" comes directly from the coefficient of p_2 in the Hirzebruch L-polynomial $\mathcal{L}_2$ and from the signature theorem, applied to an appropriate 8-manifold constructed from the 7-sphere in question.

5 The Hirzebruch-Riemann-Roch Theorem and Its Applications

We discussed Hirzebruch's signature theorem in considerable detail because it was so close to his heart and is so characteristic of his mathematics. His Riemann-Roch theorem is even more important, but since it involves the same main ingredients, characteristic classes and multiplicative sequences, we can be briefer here.

We begin by recalling the notion of the genus of a compact Riemann surface, which has both a topological and an analytic version. The topological one, going back essentially to Riemann's original paper of 1857, is a topological invariant that can be defined for any closed oriented 2-dimensional manifold (as half of the first Betti number) and is known to determine this manifold up to homeomorphism. The analytic version, also going back to Riemann, applies only when the surface has a complex structure (so is a 1-dimensional complex projective variety, or complex curve) and involves this analytic structure rather than merely the underlying topology: the genus equals the dimension of the space of holomorphic 1-forms on the curve. Its significance is shown by Riemann's inequality, which says that for any divisor $D = \sum n_i x_i$ of degree $\deg(D) = \sum n_i$ on the curve, the dimension $l(D)$ of the vector space of meromorphic functions whose divisor (= set of zeroes and poles, counted with multiplicity) is $\geq -D$ satisfies $l(D) \geq \deg(D) + 1 - g$, where g is the genus. This inequality was completed by Roch to the famous Riemann-Roch formula $l(D) - l(K - D) = \deg(D) + 1 - g$, where K is the canonical divisor.

In the later 19th century, these results were generalized to surfaces, in particular by Castelnuovo and Max Noether. In the first decades of the twentieth century, algebraic geometers studied possible ways to define the genus for an arbitrary n-dimensional projective smooth algebraic variety X. These included the *geometric genus* $g_n(X)$ (where $g_i(X)$ denotes the dimension of the vector space of holomorphic i-forms on X, so that $g_1(X)$ is the usual genus when $n = 1$), the *arithmetic genus* $\chi(X) = \sum_{i=0}^{n}(-1)^i g_i(X)$, and two further numbers denoted by $p_a(X)$ and $P_a(X)$ whose definitions we omit. It was conjectured by Severi in 1950, and proved later by Kodaira and Spencer using sheaf theory, that $p_a(X) = P_a(X) = (-1)^n(\chi(X) - 1)$. A fourth definition was proposed by John Todd in 1937 as a certain combination of the Chern numbers of X and was conjectured by him to be equal to the arithmetic genus. But it was neither known how to prove this equality nor how to describe the required combination of Chern numbers in general. The solution to both problems that Hirzebruch found used the multiplicative sequences that we have already seen. He defined the *Todd class* as the multiplicative characteristic class $\underline{\chi}_f$ corresponding to the power series $f(x) = x/(1 - e^{-x}) = 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \cdots$, so that the first components of the Todd class $\mathrm{Td}(E) = 1 + T_1 + \cdots$ of a complex vector bundle E are given by

$$T_1 = \frac{c_1}{2}, \qquad T_2 = \frac{c_1^2 + c_2}{12}, \qquad T_3 = \frac{c_1c_2}{24},$$
$$T_4 = \frac{-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4}{720},$$

where c_i denotes the ith Chern class of E. Just as before, this also defines a genus, by associating to any smooth complex variety X the value of the (top-dimensional component of) the Todd class $\mathrm{Td}(X)$ of its tangent bundle evaluated on its fundamental class. The result that Hirzebruch proved for projective algebraic varieties X, generalizing formulas that had been found in low-dimensional cases by Todd, was that this Todd genus coincides with the arithmetic genus:

$$\chi(X) = \big\langle \mathrm{Td}(X), [X] \big\rangle = \int_X \mathrm{Td}(X).$$

For $n = 1$ this gives $\mathrm{Td}(X) = 1 + \frac{1}{2}c_1(X)$ and hence $\chi(X) = \frac{1}{2}\langle c_1(X), [X]\rangle = \frac{1}{2}e(X) = 1 - g$. Thus in this case the formula reduces to the previously discussed equality of the topological and analytic definitions of the genus of a Riemann surface. To generalize the full Riemann-Roch equality, which involves a divisor (or equivalently, complex line bundle) on a Riemann surface as well as just the surface itself, we need to consider not only an n-dimensional complex variety X, but also a (holomorphic) complex vector bundle E on X of some dimension d and to find a formula for the holomorphic Euler characteristic of its sheaf cohomology groups, i.e., for the number

$$\chi(X, E) = \sum_{i=0}^{n}(-1)^i \dim H^i(X, E).$$

If X is a curve (Riemann surface) and E the line bundle corresponding to a divisor D on X, then this sum reduces to the expression $l(D) - l(K - D)$ appearing in the classical Riemann-Roch formula. The Hirzebruch-Riemann-Roch theorem gives the answer in the general case:

$$\chi(X, E) = \int_X \mathrm{ch}(E)\mathrm{Td}(X).$$

Here $\mathrm{ch}(E)$ is the Chern character of E, defined as $\sum_\beta e^{-y_\beta}$ where formally $c(E) = \prod_{\beta=1}^{d}(1 + y_\beta)$.

Hirzebruch's proof of his Riemann-Roch theorem was a characteristic combination of high-powered topological machinery and virtuoso computation. He wrote a long letter to Todd explaining his results and received a reply saying "Both Hodge (to whom I have showed your letter) and myself are very interested in this work ... I like your general expression. Incidentally, I have had to revise a long-held opinion that the Princeton School of Mathematicians despises anything in the nature of algorithmic ingenuity." The theorem was a major achievement, both in its own right and as a precursor of Grothendieck's generalization (first presented by him at the Mathematische Arbeitstagung of 1957, of which more later) and above all of the Atiyah-Singer index theorem, which is one of the key mathematical discoveries of the twentieth century and of which both the signature theorem and the HRR theorem are special cases.

At a more mundane level, the HRR theorem is a powerful tool to solve concrete problems in the interface between topology and complex algebraic geometry. Hirzebruch loved questions of this sort and in 1954 published a collection of 34 "problems on differentiable and complex manifolds," many of which continued to be studied for decades. As a typical example, Problem 14 asked for the possible Chern classes of all almost complex structures on $\mathbb{P}_n(\mathbb{C})$. This was solved some 40 years later by his student Michael Puschnigg. For instance, for $n = 4$ there are (up to complex conjugation) precisely three almost complex structures, with Todd genus 0, 1 and 1001!

6 Münster, the "Topologische Methoden," and the Work with Armand Borel

During Hirzebruch's two years at the Institute of Advanced Study he had been on leave from the University of Erlangen. On his return to Germany, he spent the year 1954–55 in Münster with a fellowship from the state of North Rhine-Westphalia, but then went back to Princeton the following year—this time not at the IAS but as an assistant professor at the university.

During his postdoctoral year in Münster he wrote his landmark book *Neue topologische Methoden in der Algebraischen Geometrie* and submitted it to the university as his "Habilitationsschrift" or habilitation thesis, which was a necessary preliminary for eligibility for a professorship. This book gave not only a description of the work he had done in Princeton, culminating in the Hirzebruch-Riemann-Roch theorem, but also a beautifully written exposition of the foundations of the whole field: the theory of bundles, sheaves, cohomology, characteristic classes, and all of the surrounding machinery. He had originally intended to publish it as a journal article, but in the end it became a book in the Springer "Ergebnisse der Mathematik" series. This book immediately gave him an international reputation, and was the "Bible" of the field for decades.

These two intermediate years between his IAS stay and his professorship in Bonn were also the period when he completed and wrote up his huge joint work with Borel. Technically speaking, their collaboration consisted of only a single paper, but it was a mammoth paper that was published under a single title "Characteristic classes and homogeneous spaces" in three parts in the same journal (American Journal of Mathematics 1958, 1959 and 1960). In this very influential paper the authors gave a systematic development of characteristic classes, based on a fusion of ideas from algebraic topology and homotopy theory with a deep analysis of algebraic groups and their associated Lie algebras and root systems, that provided the foundations for the whole field as well as for the Riemann-Roch theorem and for much of Hirzebruch's other work. The contents are best summarized by the opening paragraph of the paper:

> It is known that the characteristic classes of a real or complex vector bundle may be interpreted as elementary symmetric functions of certain variables, which are 1, 2 or 4 dimensional cohomology classes. If we consider the tangent bundle of the coset space G/U of a compact connected Lie group modulo a closed subgroup, it turns out that these variables may be identified with certain roots of G (or their squares). Our first purpose is to establish this connection between roots and characteristic classes, which is the basis of this paper, and to compute the characteristic classes of certain well-known homogeneous spaces. These results are then applied in particular to G/T (T maximal torus of G), and to other algebraic homogeneous spaces, where they lead to relations between characteristic classes, Betti numbers, the Riemann-Roch theorem and representation theory; they are also used to discuss multiplicative properties of the Todd genus and other genera in fibre bundles with G/U as fibre. As an application, we get a divisibility property of the Chern class of a complex vector bundle over an even dimensional sphere which yields some information about certain homotopy groups of Lie groups.

Hirzebruch presented the joint work with Borel in a series of lectures at the symposium "Algebraic Topology and its Applications" in Mexico City in August 1956. Here he also presented his "proportionality theorem" and its applications to the theory of automorphic forms, which he published separately in the conference proceedings. The proportionality theorem says that the Chern classes of the compact quotient of a bounded symmetric domain X by a properly discontinuous and fixed-point free group action are proportional to the Chern classes of the compact dual X', a compact algebraic manifold into which X is naturally embedded (which for the simplest case, the upper half-plane, is just the Riemann sphere). In the introduction to his paper, Hirzebruch says that a special case of the theorem can already be found "between the lines" in a paper by Igusa and that the proof he gives, simpler than his original one, is based on a suggestion by Borel.

7 Herr Professor Hirzebruch

Already at the end of his year in Münster, Hirzebruch had been invited to give a lecture at Bonn University at which to his surprise he saw many physicists and biologists. In fact, he was being "inspected" and soon afterwards received an offer of a full professorship in Bonn. (As he learned later, faculty opinions on this appointment had been divided, with some complaining that he was much too young, but the statement by one of the professors that this was a failing that time would surely cure had finally won the day.) It was not at all clear whether he should accept this appointment at a university that at the time had no great reputation in mathematics, and he was urged to stay in Princeton by many people—including Emil Artin, who however later changed his mind and himself returned to Germany. That he decided not to make his career in America was due partly to Inge's preference for living in Germany and bringing up the children there, but surely also partly because he already had the vision of helping to restore German mathematics and the credibility of Germany as a decent country.

Despite many offers of professorships at other universities, Hirzebruch never left Bonn, remaining there uninterruptedly (apart from a sabbatical year in Princeton in 1959–60 and many short stays at universities around the world) from 1956 until his retirement in 1993 and maintaining a full undergraduate teaching load even after founding the Max Planck Institute and directing it from 1980 onwards. During his years at the university, and largely under his influence, the number of full professors in mathematics increased from three to six, and he attracted such stars of pure mathematics as Jacques Tits (in 1964), Wilhelm Klingenberg (1966), Günter Harder (1969) and Egbert Brieskorn (1975). He served as dean of the Faculty of Science from 1962 to 1964, as well as taking on many other tasks in the academic community, e.g. as a principal editor of both *Topology* and *Mathematische Annalen* from 1961 until his retirement and as a very active referee for both the Deutsche Forschungsgemeinschaft (German Science Foundation) and the Alexander von Humboldt Stiftung.

Hirzebruch was an extraordinary teacher. He loved teaching at all levels and made a point of "accompanying" the same undergraduates over several years, teaching the beginning courses one year, then the second-year courses, and so on. His lectures seemed very relaxed and even leisurely, always finishing exactly on time and with the available blackboards exactly and very legibly filled. But when I once had to substitute for him during a two-week absence, and despite having his perfectly prepared notes, I had to race so hard that all the students complained about my speed, yet managed to cover only two-thirds of the material that he would have presented. The same, of course, held for his advanced lectures, in which he would seem to be introducing his subjects at such a gentle pace and in such an elementary way that his hearers often felt that they could have digested more, only to find when they tried to go through the material in detail that they had been given enough mathematical nourishment to serve them for weeks. But, as Atiyah, describing this knack, once memorably said, "rabbits do not appear out of hats unless they are put there!"

An anecdote that he liked to recount perhaps explains why or how he developed this trick of deceptive simplicity. When he was still a very young professor, he was invited to give a colloquium lecture at another German university but was warned that the level there was not very high and that he should at least start at an elementary level. So, although his goal was to give his audience an inkling of his higher-dimensional generalization of the Riemann-Roch theorem, he spent most of the hour explaining the case of curves, which had been known since the 19th century. To make doubly sure, he even spent most of his time on the case of the projective line, for which the theorem is basically empty, reducing essentially to the statement that a rational function of one variable has equally many zeros and poles. And, not to take any risks at all, he even recalled the definition of the projective line as the union of the complex number field and a point at infinity. When the lecture ended, there was dead silence, broken finally by the professor who had invited him and who obviously felt that there had to be at least one question. The question was: “This point at infinity, Herr Professor: does it really exist?”

The other part of Hirzebruch’s teaching that must be mentioned was his role as research adviser. He supervised the theses of over fifty doctoral students, including (to mention only some of those with whom I have had personal contact) Klaus Lamotke, Dietmar Arlt, Rabe von Randow, Egbert Brieskorn, Detlef Gromoll, Klaus Jänich, Karl Heinz Mayer, Wolfgang Meyer, Friedhelm Waldhausen, Winfried Scharlau, Dieter Erle, Helmut Hamm, as well as Werner Meyer, Walter Neumann, Erich Ossa, Matthias Kreck, Ulrich Karras, Alois Scharf and Karl-Heinz Knapp. I mention this last group of seven separately because it was they, together with Ulrich Koschorke (who had done his Ph.D. under Richard Palais) who formed the group of graduate students and postdocs that I became part of when I arrived myself in Bonn in 1970. It was a terrific group, but what impressed me above all was the way they did mathematics, concentrating not only on their own research and on the problems that they could discuss with their supervisor, but also organizing seminars on high-level topics like spectral sequences or Spanier-Whitehead duality that they chose themselves and at which each participant was assigned some part of the theory to study in detail and to present to the others. It was impossible to go through such a training and not become a mathematician!

8 The Arbeitstagung

Already one year after coming to Bonn, Hirzebruch got a sum of 1000 Deutschmarks from the rector of the university to organize a new conference, which he called the Arbeitstagung (literally, “work meeting”). This meeting, which has survived as an annual or bi-annual event until today, was one of his most successful and original inventions.

The first Arbeitstagung, which took place in 1957 from Saturday, July 13, until the following Saturday, had only six participants. But what a six! They were Michael Atiyah, Hans Grauert, Alexander Grothendieck, Friedrich Hirzebruch, Nicolaas Kuiper, and Jacques Tits. All of them lectured, but the dominating event was unquestionably Grothendieck’s lecture series “Kohärente Garben und verallgemeinerte Riemann-Roch-Hirzebruch Formel auf algebraischen Mannigfaltigkeiten” in which, in a total of 12 hours of talks, filling all of Saturday afternoon and Monday, Tuesday and Wednesday mornings, Grothendieck presented his generalization of the Hirzebruch-Riemann-Roch theorem. Already at this first AT the principle was established which was to give the meetings their unique flavor: it was decided only in discussions after the participants had come together who would speak, and how often, and about what.

One year later the number of speakers had doubled: Abhyankar, Bott, Grauert, Grothendieck, Hirzebruch, Kervaire, Milnor, Puppe, Remmert, Serre, Stein and Thom. Bott lectured about his periodicity theorem, already a year old but not yet known to the participants, and about applications to divisibility properties of Chern numbers, and many of the other talks were on related topics. The beginnings of topological K-theory and of the Atiyah-Hirzebruch collaboration can perhaps be traced here. And in 1962 the big excitement was the talk by Atiyah on "Harmonic spinors" (and its follow-up with the nice title "Explanations of my preceding lecture" four days later), in which the Atiyah-Singer index theorem—still in the process of being worked out in Oxford and still partially conjectural—was introduced to the "general public" for the first time. Hirzebruch gives a very lively account of these first Arbeitstagungen in [11].

In the subsequent years the Arbeitstagung became more and more popular, with the number of participants reaching well over 250 in some of the later years. The "democratic" principle was maintained, but in a very special form of "guided democracy": except for the opening talk on Friday afternoon, which was decided in advance and was almost always given by Atiyah (with the joke running that he always talked about the index theorem, but never said the same thing twice), the entire program was decided "on the spot" by the participants in three program discussions. These were strategically placed on Friday, Sunday, and Tuesday afternoons, so that no speaker ever had more than one or two days notice that he or she was going to talk (though it could often be observed that participants who had produced a result of note during the preceding 12 months came equipped with extensive notes, just in case). These discussions were occasionally heated, with people shouting out names of possible speakers and topics (the rules, not always respected, required that one had to name both, and that the nominee was not allowed either to refuse or to pick a different subject) and the chairman, who was usually Hirzebruch himself, writing them neatly on the board or, sometimes, skillfully not hearing them. There were also special social events that became part of the tradition: the Rector's party in the ballroom of the university's Poppelsdorf Castle, the boat trip on the Rhine (during which half of the participants were typically so engrossed in mathematical discussions that they forgot to look at the sights), followed by a walk up a hill in one of the nearby Rhine villages to get one's strawberry cake, and, at least in the first decades when the number of people was not too unmanageable, often a party in the Hirzebruchs' apartment in Endenicher Allee. Not surprisingly, people who had participated once usually wanted to come again, and the list of "regulars" grew from year to year.

The Arbeitstagung was always an exciting event, both because of its special organization and the endless curiosity about who would be speaking, but also because of its almost universally high level. It was very often here that one heard about a mathematical breakthrough for the first time or first became aware of a new star. (Among the début appearances that I personally remember were those of Gerd Faltings, Simon Donaldson, Frances Kirwan and Maxim Kontsevich.) As the years went by, it lost something of its original innocence and appeal, with too many participants for everyone to get to know everyone else personally and with occasional hard feelings about who had been chosen to speak, something that Hirzebruch deplored. But even so it remained for many decades one of the key events in the annual calendar of countless European and non-European mathematicians (Fig. 2).

Fig. 2 50 years of Arbeitstagung. *Left*, *top*: A. Grothendieck, M. Atiyah 1958. *Center*: R. Bott, M. Atiyah, F. Hirzebruch 1960. *Bottom*: G. Segal, M. Atiyah, J. Tits 1965. *Right*, *top*: R. Thom, F. Hirzebruch 1965. *Center*: participants of the 30th Arbeitstagung 1991 with S. Kobayashi, R. Kühnau, E. Calabi, F. Hirzebruch. *Bottom*: Hirzebruch giving the opening lecture in 2007, 50 years after the first Arbeitstagung

9 The Collaboration with Atiyah and the Invention of Topological K-theory

Hirzebruch's friendship with Michael Atiyah was one of the central axes of his life and of his mathematics. Together with Raoul Bott and Iz Singer they became an inseparable group—later known affectionately as the "Gang of Four"—during Hirzebruch's visits to Princeton, and the many discussions of the foursome during this time were important in the development of K-theory by Atiyah and Hirzebruch, of the Atiyah-Singer index theorem, and of the equivariant Atiyah-Bott-Singer index theorem [10]. Atiyah and Hirzebruch met many times in America, England and Germany, and wrote a total of nine papers together. As already mentioned, Atiyah gave the opening lecture, the only one fixed in advance, at nearly every Arbeitstagung. Hirzebruch proudly recounted that Atiyah had spoken 32 times at the 30 Arbeitstagungen that he organized, from 1957 to 1991, as well as at several of the "Arbeitstagungen of the Second Series" organized by the four directors of the MPIM every two years from 1993 onwards. Their friendship, which included also their families, lasted until Hirzebruch's death, and joint portraits of the two of them hang in both Bonn and Edinburgh.

The main period of their collaboration was from 1959–1962, during which they wrote eight joint papers (in three languages) containing the development of topological K-theory and many applications. This work, which began when they both had a sabbatical term at the IAS in 1959, was inspired on the one hand by the algebraic-geometric K-theory of Grothendieck—itself an outgrowth of his generalization of the Hirzebruch-Riemann-Roch theorem—and on the other by the Bott periodicity theorem. In fact there were two theories, a real and a complex version. Both immediately became powerful tools in topology and index theory, and they were also important as the first widely used examples of generalized cohomology theories.

The basic idea of topological K-theory is to associate to a space (finite CW-complex) X a group $K(X)$ in the complex case or $KO(X)$ in the real case, defined as the Grothendieck group of complex (resp. real) bundles over X, i.e., as the group of formal $\mathbb{Z}$-linear combinations of such bundles where $[E]$ is identified with the sum of $[E']$ and $[E'']$ whenever one has a short exact sequence of bundles $0 \to E' \to E \to E'' \to 0$ over X. A key ingredient, as already mentioned, is the Bott periodicity theorem, which Bott had just discovered. This theorem describes a periodicity—with period 2 in the complex case and period 8 in the real case—in the homotopy groups of classical groups: $\pi_k(U) = \pi_{k+2}(U)$ and $\pi_k(O) = \pi_{k+8}(O)$, where U and O denote the infinite unitary and orthogonal groups, respectively. Another formulation of this (in the complex case) is that the second loop space $\Omega^2 U$ of U is homotopy equivalent to U, or equivalently $\Omega^2 BU \sim \mathbb{Z} \times BU$, where BU is the classifying space for stable complex vector bundles. The K-group $K(X)$ of a space X can be obtained as the set of homotopy classes of maps from X to $\mathbb{Z} \times BU$, and this leads finally to a $\mathbb{Z}/2\mathbb{Z}$-graded generalized cohomology theory $K^n(X)$, and similarly to a $\mathbb{Z}/8\mathbb{Z}$-graded generalized cohomology theory $KO^n(X)$ in the real case. The realization that these groups of vector bundles constitute generalized cohomology theories, i.e., functorial maps from topological spaces to abelian groups satisfying the Eilenberg-Steenrod axioms, was a key insight that had a profound effect on the algebraic topology of the subsequent decades.

Among the important tools developed by Atiyah and Hirzebruch in the course of their study of topological K-theory was their famous spectral sequence, which they introduced for that case but which applies equally to any other generalized cohomology theory h^*. The E_2 term of this spectral sequence is the ordinary cohomology group $H^i(X, h^j(pt))$ of the space X with coefficients in the generalized cohomology of a point, and the spectral sequence converges to $h^{i+j}(X)$.

Hirzebruch and Atiyah planned to write a book-length exposition of topological K-theory, but in the end never did, though Atiyah later wrote such a book alone based on a course of lectures that he gave at Harvard in 1964. The theory quickly became a basic tool of algebraic topology, with many later important applications, including two very famous ones by J.F. Adams: a simple proof of the Hopf invariant one problem, and an upper bound for the number of linearly independent vector fields on spheres.

The starting point of their joint work on K-theory is described by Hirzebruch in his report [11] on the early Arbeitstagungen: "At the ICM 1958 Atiyah mentioned to me that Grothendieck's method gives the integrality of the Todd genus of a stable almost complex manifold X by embedding X in an even-dimensional sphere. Discussion showed that one can prove this way that all RR-numbers are integers. We were on our way to the topological K-theory and the 'differentiable Riemann-Roch theorems'." The paper referred to here is their first joint one, in which a new proof is given that certain divisibility properties for Chern numbers of complex algebraic varieties that followed from the Hirzebruch-Riemann-Roch theorem were in fact true for all almost complex manifolds. This question, raised by Hirzebruch, had already been answered largely by work of Borel-Hirzebruch and Milnor using cobordism theory and the Adams spectral sequence, but the new proofs, based on the emerging K-theory, were completely different and much more natural. Similar ideas led to their second paper (the one in French) giving conditions for the minimum dimension of a sphere into which certain manifolds can be embedded. Several other of their joint papers gave further important applications of K-theory and index theory to concrete problems. One, for instance, showed that the Hodge conjectures are false for integer cohomology (the case of rational cohomology is of course still open), while another was related to Hirzebruch's earlier discovery of a relation between Steenrod squares and Todd polynomials.

Their last joint paper was written in 1970, several years after the others, and combined in a beautiful way Hirzebruch's theory of genera with Atiyah-Singer index theory. One defines the "A-hat" class as the multiplicative characteristic class associated in the way explained above with the power series $\frac{x/2}{\sinh(x/2)} = 1 - \frac{x^2}{24} + \cdots \in \mathbb{Q}[[x]]$. Since this is an even power series, the corresponding characteristic class can be written as a polynomial in the Pontryagin rather than Chern classes and hence gives a cohomology class $\widehat{A}(TM) \in H^*(M;\mathbb{Q})$ for any smooth orientable manifold M. Borel and Hirzebruch proved that the corresponding genus $\widehat{A}(M) = \langle \widehat{A}(TM), [M] \rangle$ is always integral if M is a spin-manifold (one with vanishing second Stiefel-Whitney class), and Atiyah and Hirzebruch proved that it vanishes if M admits a non-trivial S^1-action. The integrality comes from index theory, since from the Atiyah-Singer index theorem one obtains an interpretation of $\widehat{A}(M)$ as the index of the Dirac operator acting on spin-bundles over M. This work later took on a new life in connection with the discovery by Ochanine and others of the elliptic genus, in which Hirzebruch took a lively interest and which will be described later.

Although Atiyah and Hirzebruch wrote no further joint papers after 1970, they always remained in close mathematical (as well as of course personal) contact, and a conjecture that Hirzebruch made relating a topological invariant of cusp singularities of higher-dimensional Hilbert modular varieties to special values of certain number-theoretical L-functions was the inspiration for later papers of Atiyah with Patodi and Singer on the index theorem for manifolds with boundary and with Donnelly and Singer to prove Hirzebruch's conjecture.

Fig. 3 F. Hirzebruch, newly appointed honorary fellow of the Royal Society of Edinburgh in 2010, with M. Atiyah

10 Founding an International Center: The First Attempts

Already in 1955 during his second stay in Princeton as assistant professor, Hirzebruch had met Reimar Lüst (later the third president of the Max Planck Society) and had communicated his enthusiasm for the IAS. He was convinced that this model—a non-university institution that had few permanent members, but invited both recognized senior and promising young mathematicians for shorter or longer stays, in a relaxed atmosphere and free of academic duties and worries—was a wonderful way to promote mathematical research, and his goal from then on was to set up an institute of similar type in Germany. The awareness that the impetus for the creation of the IAS had been the exodus of both Jewish and non-Jewish scientists from Germany in the 1930's, and the conviction that Germany had a debt to repay, certainly also contributed to his determination.

As we have already told, Hirzebruch had received the offer of a chair in Bonn shortly before leaving for his second Princeton stay in 1955, and when he and his new young family decided to return to Germany and accept this position, he immediately started working towards his goal. With invitations to Nicolaas Kuiper and Raoul Bott as guest professors and the initiation of the iconic Arbeitstagung, the first small but very important steps were taken.

In 1958, the newly founded European Atomic Energy Community (EURATOM) had plans to found a European university-level institution and, within that framework, also a European Institute for Advanced Study for Mathematics (EUROMAT). Each EURATOM country was invited to send two representatives to the first planning meeting, those from Germany being Hellmuth Kneser from Tübingen and Wilhelm Süss, who was then the director of the Mathematics Research Institute in Oberwolfach. Since Süss was very ill, Hirzebruch was asked to replace him and agreed. The meeting took place and its participants were urged to present the plans to the ministries of their respective countries, which Hirzebruch did, at the same time also working hard to get more government support for Oberwolfach, of which Kneser had assumed the direction after Süss's death the same year.

However, in July of the same year Hirzebruch learned that a French research institute in mathematics and physics, the IHES, had been founded by the industrialist and mathematician Léon Motchane, with himself as first director and with financial contributions from French and Italian companies as well as from EURATOM. It was clear that the EUROMAT proposal was doomed and Hirzebruch immediately gave it up. Instead, still in the same month, he wrote up a proposal for turning Oberwolfach from a conference center into a full-fledged research institute. The ministry of interior affairs, to whom this proposal was addressed, was very favorable. Further discussions with Kneser and with Paul Hübinger from the ministry followed, during which it emerged that the optimal solution would be the creation of a new Max Planck Institute for Mathematics with two seats: the original one in Oberwolfach, that would serve as a meeting and conference place, and the other one—for practical reasons—in Freiburg, where the guest researchers and their families would live. A new "Society for mathematical research" that would serve as a link between political and research organizations was founded in June 1959 and immediately began discussions with the Max Planck Society, and already in December 1959, during Hirzebruch's sabbatical leave from Bonn to Princeton, he got a formal letter asking whether he would be willing to conduct the negotiations for the creation of the new institute and to serve as its first director if the application was approved. This letter explicitly mentioned the guiding principle of the MPS that each new institute should be founded around the science and the visions of one person, not just in a chosen field. Despite some reservations, in particular about the impact of the new job on his teaching activities, he agreed, writing that the new institute should have about 15 visiting positions and that the visitors' fields should vary. On January 6, 1960, the Senate of the Max Planck Society approved the project of setting up an MPIM with Hirzebruch as director, and in May 1960 a major German newspaper reported that Adolf Butenandt, who had just been named the new President of the MPS, had announced the Society's plan to found two new MPI's in the humanities (!), of which one was to be in mathematics.

But it was not to be. As part of the process of founding the projected institute, the MPS asked for recommendation letters from respected external mathematicians, and two of these reports, from Siegel and from Courant, were negative—for the wrong reasons. Siegel thought that Hirzebruch's main field, abstract algebraic geometry, was a passing fad and like the rest of modern abstract mathematics had no future. Apart from this very wrong prediction, Siegel seems not to have captured that in any case the future MPIM would aim to pursue fields covering a large part of mathematics, not just the research interests of its director. The objections of Courant, who was an admirer of Hirzebruch, were different: He considered it a crime to rob students of such a splendid lecturer. This, too, was unfounded since Hirzebruch had written clearly in his application of his intention to continue teaching his full load at the university even if the MPIM was founded (and indeed did so when many years later it finally was). But Courant was also an opponent of non-university mathematical institutes in general, and like Siegel disliked the trend to the abstract in mathematics. The two reports, in particular Courant's, were decisive, and in November 1960 the MPS, very confused by the conflicting advice that they had received and in any case not knowing quite how to handle an institution centered on visitors rather than laboratories, formally turned down the application for the new institute.

11 The Sonderforschungsbereich Theoretische Mathematik

But Hirzebruch's determination to fight for establishing an international research center in Germany continued, and there were new political currents favourable to this idea. In 1969, there was a big initiative from the Deutsche Forschungsgemeinschaft (German Research Foundation) to set up "Sonderforschungsbereiche" (Collaborative Research Centres) within many German universities. These were meant to foster multidisciplinary research around an already existing excellence kernel, beyond the normal university programs or possibilities. With primarily R. Leis and H. Unger from the more applied directions and Hirzebruch for the more theoretical side, the Faculty of Sciences had prepared an application for an SFB, and Hirzebruch was chosen to present this application to the DFG. In fact, he came back with *two* SFB's, one each in the more "pure" and more "applied" fields: the SFB 40 "Theoretical Mathematics" under his leadership and the SFB 72 "Approximation and Mathematical Optimization" under that of Leis and Unger.

In the following years, with the help of Silke Suter, who assisted him first part time as a graduate student and then full time, he slowly built up the scientific and administrative structures of the new center and developed the model that would later serve as the starting point for the Max Planck Institute. The SFB 40 bloomed, with the Arbeitstagung and an already extensive guest program as its flagships. Among the early visitors were many eminent mathematicians, including Robert Langlands, Armand Borel, George Mackey, Pierre (Peter) Gabriel and Serge Lang. Many of them also lectured at the university, enriching the students' program. It was sometimes said at the time that there were many mathematicians in the world who knew only two words of German: "Arbeitstagung" and "Sonderforschungsbereich."

Through the many visitors to the Sonderforschungsbereich, and later also to the Max Planck Institute, Hirzebruch became more and more closely involved with the mathematics and mathematicians of other countries. His work for and with the countries of Eastern Europe, and then with Israel, will be described below in some detail, but here we mention Japan in particular because it was precisely the flexible structure of the visitors' program of the SFB that permitted this relationship to flourish. Hirzebruch had first visited Japan in 1972, on the invitation of his friend Kodaira, who was then the dean of the University of Tokyo, to give IMU lectures, and had met many of Kodaira's brilliant students whom he later invited to Bonn. In the following years, the stream of visitors from Japan steadily increased, sometimes so numerous that a whole seminar would be held only in Japanese. There is an old MPI photo that shows a group of more than 20 Japanese who were visiting at the same time. This was due in particular to the fact that in those times Japanese mathematicians could only go abroad for a total of two years in their academic career, and since going abroad with their families meant that they practically moved their whole household, they were happy to get invitations for two years continuously, something that the SFB and later the MPI could do but that few universities or other research institutions could. Because Hirzebruch did so much for the Japanese connection, he later received the prestigious Seki prize from the Japanese Mathematical Society and the Order of the Holy Treasure from the Japanese Emperor.

12 Building Bridges to the East

One of the aftereffects of World War II of which Hirzebruch was most painfully aware was the Cold War division of Europe into East and West, and throughout his life he did what he could to counteract this split. This applied above all to "East Germany" (more properly the German Democratic Republic, or GDR), but also to Russia, Poland and the other countries behind the Iron Curtain.

Hirzebruch's involvement with East Germany was marked by a striking symmetry: in 1962 he was the chairman of the Deutsche Mathematiker-Vereinigung (German Mathematical Society) when it was broken up into an only West German society and a new "Mathematical Society of the GDR," and it was he again who held the same office in 1990 after the fall of the Wall and who was able to preside over the reunification after 28 years of the two estranged siblings. (See [3].)

During the beginning of his first chairmanship, the DMV still had both East and West German members, but the Wall had already been built (in 1961) and there was no place where the members from the GDR, from West Berlin and from West Germany could come together for their meetings. Hirzebruch's elegant solution to this was to simply hold the entire program twice, in East and in West Berlin. After the society had been split, he could do nothing more in his DMV role, but during the entire period until reunification he was very active in many other ways. His main concern was the gradual freezing of contacts, which he tried to prevent by inviting GDR mathematicians to the Arbeitstagungen, by many personal visits and lectures in universities and academies there—the Hirzebruch family remembers him often shopping for presents in Bonn the morning before boarding the train—and by individual invitations to the SFB and later to the MPI. Mostly these invitations could not be realized, but there was some modest success with the Arbeitstagung invitations: among the people who came were Herbert Kurke, Wolfgang Vogel, Eberhard Zeidler, Thomas Friedrich, Jens Franke, Jürgen Leiterer and Werner Müller. Sometimes mathematicians who came as visitors remained in the West and were helped by Hirzebruch to find positions. During the years when many West Germans considered the country beyond the Wall as too forbidding and too complicated to visit, he never let himself be discouraged. In the words of Eberhard Zeidler, "er hat den Faden nie abreißen lassen"—he never let the thread be severed.

When Hirzebruch became president of the DMV for the second time 28 years later, the situation was wildly different. The wall had just fallen, and during the first meeting over which he presided he could proudly announce the reunion of the two mathematical societies into one. But then following the reunification of Germany there was a new problem: the reintegration of the former East German mathematicians into the new system. This turned out to be a gigantic task, but one for which Hirzebruch was especially suited, since he was one of the few people known and trusted by both East and West. The smaller problem concerned the DMV itself, in particular since the Mathematical Society of the GDR had had many more school teachers and industrial mathematicians among its members than was customary in the West. But the major problem was that a very large number of positions, including assistant positions, had been permanent in the East and suddenly no longer were. In particular, the Karl Weierstrass Institute of the GDR Academy of Science employed nearly 200 mathematicians, many in pure research positions, whereas in West Germany, with a three times larger population, there were practically no positions of this sort apart from the four directors of the Max Planck Institute. For a long time, Hirzebruch traveled every week to Berlin where he tried—together with the Max Planck Society, the Fraunhofer Society, and members of the federal and state governments—to find solutions for the GDR scientists and working groups. There were many different answers: some of the mathematicians were well known internationally and easily got professorships or research positions in Germany or abroad, others were near retirement and could obtain temporary positions to bridge the remaining years, and yet others found employment in schools or industry. Special programs were set up, for example 28 working groups in different fields that were financed for five years by the Max Planck Society and attached to a university, in the hope that the groups could later be integrated into the university. But there was not a solution for everyone and some remained jobless. Hirzebruch was never reconciled with this situation and suffered sleepless nights where fate had been unkind.

Today these problems are a thing of the past and there is hardly any difference between East and West. With some 20 new Max Planck Instiutes in the new federal states, with vivid scientific exchange and easy access by train or airplane, it seems that the only remaining distances are geographical.

Another major concern of Hirzebruch during the Cold War years was to keep contact with mathematicians from the countries behind the Iron Curtain. Every year, he invited leading mathematicians from the Soviet Union and other Eastern European countries to the Arbeitstagung. With the exception of the year 1967, when no fewer than five Soviet mathematicians (Anosov, Manin, Postnikov, Shafarevich and Venkov) were able to come, these invitations were nearly always declined, but Hirzebruch later learned that they had nevertheless helped the invitees by making the Soviet authorities aware of their international visibility. Whenever possible, Hirzebruch accepted invitations to Moscow and Leningrad, mostly by the Russian Academy of Sciences, where he gave lectures, participated in conferences, and above all met Russian colleagues and exchanged news about scientific advancements. He was elected a foreign member of the Academy of Science of the USSR in 1988, received the Lobachevsky Prize in 1990 and the Lomonosov Gold Medal in 1996. He was also actively involved in the exchange with the Euler International Mathematical Institute in St. Petersburg, which was formed in those years as part of the Steklov Institute and developed many relations to Berlin and Germany. Eventually, of course, the situation changed radically, and today it is impossible to walk through the corridors of the MPI and not hear Russian spoken.

Especially in the years after the fall of the Iron Curtain, Hirzebruch was also very engaged in supporting mathematics in Poland. In 1990, as president of the European Mathematical Society, he wrote a very strong letter of support for the Banach Center in Warsaw, which was struggling financially and politically. This led to the founding of a Banach Center Council in 1993, with Hirzebruch as its chairman for the first eight years, during which he visited the country yearly and was indefatigable in his support at every level. A very personal account of the story can be found in the article by Stanisław Janeczko in [16]. His contribution was greatly appreciated and was honored in 1998 by an international conference on the occasion of his 70th birthday and in the following year by the award of the Stefan Banach Medal of the Polish Academy of Sciences, of which he had become a member in 1997. A second birthday conference in his honor, which he planned to attend until prevented by an accident, was held in Warsaw just before his death in May 2012.

13 "More and More Number Theory in Topology"

The slogan "More and more number theory in topology" that I have taken as the title of this section is due to Hirzebruch, who proposed it "as a theme (familiar to most topologists) under the general title 'Prospects of mathematics' " in his lovely survey paper [1]. Of course his work had already had several points of contact with number theory, for instance through the appearance of Bernoulli numbers in his work on genera and the signature theorem or, at a deeper level, through his proportionality principle, which was used by him and others to compute dimensions of spaces of automorphic forms. But starting at the end of the 60's the connections grew much closer. This arose in particular from his work on singularities,[1] plumbing constructions and exotic spheres, which were treated in a course he gave and published as a joint Springer Lecture Notes with K.H. Mayer in 1968. In these notes he discussed the beautiful singularities

$$X_{\mathbf{a}} = \bigl\{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \bigm| z_0^{a_0} + \cdots + z_n^{a_n} = 0\bigr\} \quad \bigl(\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{N}^{n+1}\bigr)$$

that had been studied by his student Egbert Brieskorn, who had discovered in particular that the 7-dimensional varieties $X_{2,2,2,3,6k+1} \cap S^9$ give an explicit realization of the 28 exotic structures on S^7 found by Kervaire and Milnor as k ranges over all integers modulo 28. In this connection, Brieskorn had shown that the signature of the smooth complex n-dimensional variety $X_{\mathbf{a}}(\varepsilon)$ obtained by replacing "$=0$" by "$=\varepsilon$" in the definition of $X_{\mathbf{a}}$ is given by

$$\begin{aligned}\operatorname{sign} X_{\mathbf{a}}(\varepsilon) = \sharp\Bigl\{\boldsymbol{t} \in \mathbb{Z}^{n+1} \Bigm| 0 < t_i < a_i,\ r < \frac{t_0}{a_0} + \cdots + \frac{t_n}{a_n} < r+1 \text{ with } r \text{ even}\Bigr\} \\ - \sharp\Bigl\{\boldsymbol{t} \in \mathbb{Z}^{n+1} \Bigm| 0 < t_i < a_i,\ r < \frac{t_0}{a_0} + \cdots + \frac{t_n}{a_n} < r+1 \text{ with } r \text{ odd}\Bigr\}.\end{aligned}$$

This formula was the starting point for my own relationship with Hirzebruch. I had read his notes as a second-year graduate student at Oxford, and by elementary manipulations found that Brieskorn's formula was equivalent to the identity

$$\operatorname{sign} X_{\mathbf{a}}(\varepsilon) = \frac{(-1)^{n/2}}{N} \sum_{\substack{0<j<2N \\ j \text{ odd}}} \cot\frac{\pi j}{N} \cot\frac{\pi j}{a_0} \cdots \cot\frac{\pi j}{a_n},$$

where N is any positive multiple of the exponents a_i. The presence of the cotangent function in this expression was strongly suggestive of the equivariant signature and Riemann-Roch theorems, and I wrote to Hirzebruch to show him my formula and also to ask about the possibility of coming to Bonn to continue my doctoral studies under

[1] A wonderful exposition of Hirzebruch's work on singularities was given by Brieskorn in [9].

his supervision, since my Oxford supervisor Michael Atiyah had left for Princeton. Hirzebruch, to whom Atiyah had also written with the same suggestion, invited me to Bonn in May 1970 (a memorable occasion for me, as I met both him and my future wife, then Silke Suter, on the same day), spent an entire afternoon discussing these matters with me and Brieskorn, who was just visiting Bonn from Göttingen, and invited me to come both to the Arbeitstagung in June and to Bonn as his Ph.D. student in the Fall.

The next phase of Hirzebruch's involvement with number theory was partly in tandem with me. We found that the cotangent formula I had sent to him could indeed be obtained by a suitable application of the equivariant signature theorem, and more generally that one could find many further topological situations, involving specific low-dimensional manifolds with explicit group actions, where the signature or other topological invariants could be computed both directly from their definitions and also by applying various index theorems, yielding wildly sophisticated proofs of elementary number-theoretical identities. Hirzebruch presented these results in lectures that he gave in Japan and then in a course in Bonn that we published as a joint book "The Atiyah-Singer Index Theorem and Elementary Number Theory" in the nicely named series "Publish or Perish" of Michael Spivak, who was then a visitor at the SFB. This work connected topology in sometimes surprising ways with Dedekind sums, class numbers, periodic continued fractions, and Markoff numbers, all very venerable themes of classical number theory. A nice result that combined the first three of these themes, based on a theorem of Curt Meyer expressing the value at $s = 1$ of a certain L-series in terms of Dedekind sums, relates the class numbers of real and quadratic fields to continued fractions: if $p > 3$ is a prime congruent to 3 (mod 4) for which the real quadratic field $\mathbb{Q}(\sqrt{p})$ has class number 1, and if $(b_1, \dots, b_r)$ denotes the minimal period of the continued fraction with negative signs of $\sqrt{p}$ (see next section), then the class number $h(-p)$ of $\mathbb{Q}(\sqrt{-p})$ equals $\frac{1}{3}\sum_{i=1}^{r}(b_i - 3)$. A planned joint paper presenting this and related results unfortunately never got written, though the class number formula was stated in papers of each of us separately.

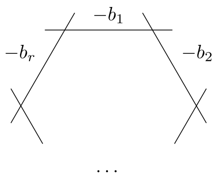
14 Hilbert Modular Surfaces and the Resolution of Their Cusp Singularities

Among the various specific low-dimensional manifolds that Hirzebruch and I had studied for our applications of index theorems to number theory were torus bundles over a circle. These led directly to the far more exciting second phase of Hirzebruch's life as a number theorist, his study of the geometry of Hilbert (or Hilbert-Blumenthal) modular surfaces, to which I now turn. This investigation had three main parts: the resolution of the singularities of Hilbert modular surfaces at their cusps, the computation of the numerical invariants of these surfaces and its application to determining their type (rational, $K3$, elliptic, or general type, or equivalently Kodaira dimension $-\infty$, 0, 1 or 2) in the classification of algebraic surfaces, and the study of the intersection behavior of the modular curves lying on these modular surfaces. I will describe all three briefly.

Hilbert modular surfaces are a generalization of the modular curve $\mathfrak{H}/\mathrm{SL}(2, \mathbb{Z})$ ($\mathfrak{H}$ = complex upper half-plane) from the classical theory of modular forms. To a real quadratic field K one associates (at least in the simplest case; there are many variations) the quotient $X_K = \mathfrak{H}^2/\mathrm{SL}(2,\mathcal{O})$. Here $\mathcal{O} = \mathcal{O}_K$ is the ring of integers of K and $\mathrm{SL}(2,\mathcal{O})$ is the group of 2×2 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with coefficients in $\mathcal{O}$ and determinant 1, acting on $\mathfrak{H}^2$ by $(z_1, z_2) \mapsto (\frac{az_1+b}{cz_1+d}, \frac{a'z_2+b'}{c'z_2+d'})$, where $x \mapsto x'$ denotes the Galois conjugation of K over $\mathbb{Q}$. This surface has a natural compactification by the addition of h_K (= class number of K) points called the cusps. A neighborhood of the "cusp at infinity" is the quotient of the set $W_C = \{(z_1, z_2) \in \mathfrak{H}^2 \mid \Im(z_1)\Im(z_2) > C\}$, where C is a sufficiently large constant, by the group of triangular matrices $\left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right)$ with $a \in \mathcal{O}^*$ and $b \in \mathcal{O}$. For other cusps or other Hilbert modular surfaces (e.g. those obtained from subgroups of $\mathrm{SL}(2,\mathcal{O})$ of finite index) one has more generally the quotient $W_C/G(M,V)$, where $G(M,V) = \left\{\left(\begin{smallmatrix} \varepsilon & \mu \\ 0 & 1 \end{smallmatrix}\right) \mid \mu \in M, \varepsilon \in V\right\}$ for some sublattice $M \cong \mathbb{Z}^2 \subset K$ and some infinite cyclic group of totally positive units V acting on M. The problem is then to find a smooth open surface $Y_C(M,V)$ containing a divisor consisting of smooth curves with normal crossings whose complement is isomorphic to $W_C/G(M,V)$.

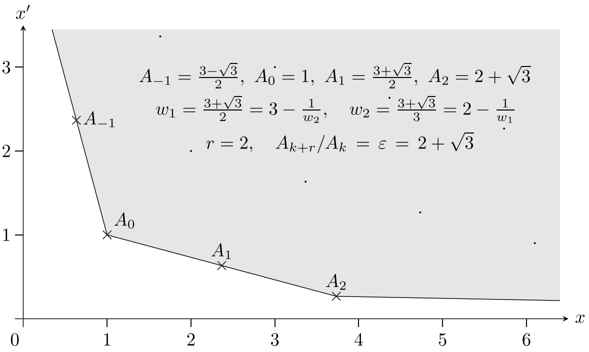
The solution of this problem was in some sense an outgrowth of Hirzebruch's doctoral thesis in which, as we saw, he had studied the resolution of 2-dimensional singularities and had shown that they are sometimes described number-theoretically in terms of continued fractions, as in the example for the singularity $w = \sqrt[n]{z_1^q/z_2}$ that we described earlier. (The necessity of using continued fractions with minus signs, rather than the traditional ones with plus signs, was one of his insights.) The solution of the problem for the cusp singularities turned out to be related. To the cusp labeled by (G,V) as above one associates (by taking $w \in K$ equal to the ratio of the two elements of a suitable $\mathbb{Z}$-basis for M) a *periodic* continued fraction expansion whose partial quotients b_i are uniquely determined up to cyclic permutation by the pair (M,V). Hirzebruch then showed that the resolution of the singularity is given by the corresponding cycle of rational curves, with self-intersections $-b_i$ just as for the finite continued fractions that had occurred in his thesis:

$$w = b_1 - \cfrac{1}{\ddots - \cfrac{1}{b_r - \cfrac{1}{b_1 - \cfrac{1}{\ddots}}}},$$



He thus made spectacular amends for the wrong opinion stated there by showing not only that cycles *can* occur in the resolution of 2-dimensional singularities, but that a major problem in number theory could be solved using precisely these forbidden cycles!

We will describe here the way this cusp resolution works, because it is both a beautiful piece of mathematics and a beautiful example of Hirzebruch's seamless fusion of ideas from topology, algebra, analysis, geometry and number theory. We embed M as a lattice in $\mathbb{R}^2$ via $x \mapsto (x, x')$. Then the boundary of the convex hull of $M \cap \mathbb{R}^2_{>0}$ consists of the line segments joining the images of countably many elements $A_k \in M$ ($k \in \mathbb{Z}$), of which any two successive ones form a $\mathbb{Z}$-basis of M and any three successive ones are related by $b_k A_k = A_{k-1} + A_{k+1}$ for some integer $b_k \geq 2$, with at least one $b_k \geq 3$.



The action of a generator ε of V by multiplication on M sends A_k to A_{k+r} for some $r > 0$ and all k, so the b_k are periodic ($b_{k+r} = b_k$; r is in fact the minimal period of the b_k multiplied by the index of V in the group of all totally positive units preserving M). Then the quotients $w_k = A_k/A_{k+1}$ satisfy $w_k = b_k - 1/w_{k+1}$ and hence all are given by periodic continued fractions like that of $w = w_1$ above, with periods related by cyclic permutations. The local ring $\mathcal{R}$ of the singularity consists of $G(M, V)$-invariant holomorphic functions $F(z_1, z_2)$ on W_C for sufficiently large C. Every such function, and in particularly every Hilbert modular form, has a Fourier expansion of the form

$$F(z_1, z_2) = c(0) + \sum_{\nu \in M^\vee,\, \nu \gg 0} c(\nu) e^{2\pi i(\nu z_1 + \nu' z_2)},$$

where $M^\vee = \{\nu \in K \mid \mathrm{tr}_{K/\mathbb{Q}}(\nu M) \subseteq \mathbb{Z}\}$ is the dual of M (e.g. $M^\vee = \frac{1}{\sqrt{D}}\mathcal{O}$ for $M = \mathcal{O}$, where D is the discriminant of K) and where $c(\varepsilon\nu) = c(\nu)$ for all ν. For each $k \in \mathbb{Z}$ we can write $F(z_1, z_2)$ as a power series $\sum c(\nu) u_k^{\mathrm{tr}(A_k \nu)} v_k^{\mathrm{tr}(A_{k-1}\nu)}$ in the coordinates $(u_k, v_k) \in \mathbb{C}^{*2}$ on $\mathbb{C}^2/M$ defined by $2\pi i \binom{z_1}{z_2} = \begin{pmatrix} A_k & A_{k-1} \\ A'_k & A'_{k-1} \end{pmatrix} \binom{\log u_k}{\log v_k}$. This identifies $\mathcal{R}$ with a subring of the product of r copies of the ring of somewhere convergent power series in two variables:

$$\mathcal{R} \cong \big\{ (F_k)_{k \ (\mathrm{mod}\ r)} \in \mathbb{C}\{u, v\}^{\mathbb{Z}/r\mathbb{Z}} \;\big|\; F_k(u, v) = F_{k+1}\big(u^{b_k} v_k, u_k^{-1}\big) \text{ for all } k \big\}.$$

Since each point on the surface is near the origin in at least one coordinate system (u_k and v_k are bounded by 1 in absolute value if the point $(\Im(z_1), \Im(z_2)) \in \mathbb{R}^2_{>0}$ belongs to the closed cone spanned by A_{k-1} and A_k), this description of the local ring corresponds geometrically to a resolution of the singularity by a cycle of rational curves S_k as above, where S_k is given by the equation $v_k = 0$ in the kth coordinate system (u_k, v_k) and by $u_{k+1} = 0$ in the $(k+1)$st coordinate system, and where the intersection point of the curves S_{k-1} and S_k corresponds to the origin $(u_k, v_k) = 0$ in the kth coordinate system. The self-intersection of S_k is $-b_k$ because the divisor of the meromorphic function represented by u_k in the kth coordinate system contains S_{k-1}, S_k and S_{k+1} with multiplicities b_k, 1 and 0, respectively, and has intersection number 0 with S_k.

Hirzebruch's resolution of the cusp singularities was one of the first non-trivial examples of what are now called toroidal embeddings and toroidal compactifications, and was one of the motivations for the development of this theory by Mumford, Satake, and others. Another offshoot of the cusp resolution was a formula that I found for the decomposition of the Dedekind zeta function of K and that was generalized by Shintani to arbitrary totally real number fields.

The second problem that Hirzebruch tackled was the determination of how the Hilbert modular surfaces fit into the Enriques-Kodaira rough classification of surfaces. A key step here was the calculation of the numerical invariants of the surface, which in turn had three ingredients: the cusp resolution, a formula for the Euler characteristic in terms of $\zeta_K(2)$ (the value at $s=2$ of the Dedekind zeta function of the quadratic field) using a Gauss-Bonnet theorem for non-compact manifolds due to Günter Harder, whom Hirzebruch had brought to Bonn as a colleague, and the signature theorem. All Hilbert modular surfaces belong to one of four classes in the classification (rational, K3, non-K3 elliptic fibrations, or general type), and Hirzebruch succeeded in classifying all of them, doing the cases when K has prime discriminant with Van de Ven and the case of general discriminant with me. The result for prime discriminants is particularly attractive: X_K is rational for the first three values 5, 13 and 17, is a K3 surface for the next three values 29, 37 and 41, is a non-K3 elliptic fibration for the next three values 53, 61 and 73, and is of general type for all larger prime discriminants. For arbitrary discriminants one finds that there are precisely 10, 9 and 13 surfaces having the three special types.

Hirzebruch's last big contribution to the theory of Hilbert modular surfaces concerned the study of certain curves on these surfaces and their intersection behavior, and involved another beautiful interplay between topology and number theory. This investigation was carried out jointly with me, and for me remains perhaps the most exciting collaboration I have ever been involved with. For each positive integer N one defines a curve T_N on X_K as the union of all points satisfying equations of the form $az_1z_2+\nu z_1+\nu' z_2+b=0$ with $a,b\in\mathbb{Z}$, $\nu\in\mathcal{O}^\vee$ and $ab-\nu\nu'=N/D$. Hirzebruch found a geometric way to describe their intersection points, and together we were able to use this to get an exact formula for the finite part of the intersection numbers, based on the theory of complex multiplication and involving some fairly serious algebraic number theory. Then using his resolution of the cusps we could also compute the intersection numbers of the corresponding curves T_N on the resolved and compactified surface. The high points were the realization that these numbers were the Fourier coefficients of modular forms (this involved recognizing the modular nature of the generating function of class numbers of imaginary quadratic fields, which I did and which later turned out to be the first example of what are now called mock modular forms) and our main theorem identifying the relevant part of the cohomology of the Hilbert modular surface with a certain space of modular forms of weight 2. This theorem was a prototype for many later generalizations in which the generating series of some geometric invariants turn out to be modular forms. An amusing story in this connection is that at one point in this study a letter from Serre to Hirzebruch, asking whether he could explain the numerical coincidence between a certain Betti number and the dimension of this space of modular forms, crossed in the post a letter from Hirzebruch to Serre in which he described our work and answered this very question!

Finally, I should mention again Hirzebruch's conjecture on the signature defect and its relation to special values of L-functions for higher number fields, which was later solved by Atiyah–Donnelly–Singer and independently by Werner Müller.

Hirzebruch wrote up the whole theory in a long article in *L'Enseignement Mathématique* that also appeared as a monograph, and then later in a second book with Gerard van der Geer based on a joint course that they had given in Montréal. This developed into a close friendship with van der Geer, with whom he later organized many meetings in the mountain village of Alpbach in the Austrian Tirol, where the two of them and a group of undergraduates would spend a fortnight alternating between intensive mathematical and intensive hiking activities.

15 The Founding of the Max Planck Institute for Mathematics

We have already told of Hirzebruch's first abortive attempts to found a Max Planck Institute of Mathematics, and of the successful founding of the Sonderforschungsbereich Theoretische Mathematik instead. But the lifetime of a Sonderforschungsbereich was normally 6 or 9 years, and could certainly not exceed 15. So already in 1977/78 Hirzebruch discussed with the third Max Planck president, Reimar Lüst, the prospects of a new attempt at establishing an MPIM. This time the auspices were more favourable: not only was there a very successful SFB to build upon and which seemed to meet a real demand, but Hirzebruch had also kept his full teaching load at the university, and moreover, the costs of the institute would be modest since there were no guest researchers to transfer into permanent positions. In a letter to Lüst that was only three and a half pages long, Hirzebruch outlined the structure and functioning of the future institute, as well as a program for the period of transition of the Sonderforschungsbereich into the MPIM. His letter formed the basis both for discussions inside the MPS and later, in 1979, for a meeting organized by the Alexander von Humboldt Foundation, to which Saunders McLane and other prominent mathematical scientists from different fields of mathematics were invited and where Reimar Lüst was present. The discussion there centered on the situation of mathematics in Germany, and soon Hirzebruch realized that the meeting was meant as a launching pad for the project of founding an MPI in mathematics. Indeed, in the following year, the senate of the MPS decided to found the MPIM. There followed a transition period of three years, during which the financing of the visitors's program was covered by the SFB and the MPIM in the ratios 75:25, 50:50 and 25:75, respectively. By 1984 the MPIM had taken up its full activities and the institute had moved from its old quarters in Beringstraße to a rented office building in Beuel on the other side of the Rhine (Fig. 4). The full story is recounted by Norbert Schappacher in [2].

According to plan, there were very few permanent positions—in the beginning only Hirzebruch full time, myself half-time (the other half being spent in Maryland), and Harder as an External Scientific Member. Following the Princeton model, most of the guests were young researchers, mainly post-docs, from all over the world, with the Japanese and the Russian groups standing out in size and influence, as we have already described. As the number of visitors grew, so did the scientific and non-scientific personnel. I became a full member in 1984 and Harder in 1991. We were

Fig. 4 The president of the Max Planck Society, Reimar Lüst, with F. Hirzebruch at the inauguration of the Max Planck Institute for Mathematics on June 24, 1982

joined by Yuri Manin in 1993 and Gerd Faltings in 1994, so that by the time that Hirzebruch retired in 1995 the institute had a Director and four Scientific Members. Since none of the four of us felt capable of assuming his role, we suggested to the Max Planck Society that we should all share the title and duties of director, with the role of managing director rotating among us on a two-year basis, and this has been the model ever since, with Werner Ballmann and Peter Teichner later joining the institute as the successors of Harder and Manin, respectively. In the course of time, there were a few further long-term positions, most notably the special professorship that was created for Matilde Marcolli (gauge theory, mathematical physics, non-commutative geometry), who joined the MPIM from 2000 and 2008, won the Leibniz prize for the Institute, and was indefatigable in arranging research activities and in taking care of young students from the IMPRS graduate school program that started in 2003. The Institute now also has three External Scientific Members, Werner Nahm since 2006 and Sergei Gukov and Stefan Müller since 2010.

In 1999, the MPIM moved from Beuel to its current premises in the former "Post-carré" in the center of Bonn, creating much more space for offices, seminar rooms and library and making it easier to commute to the University. The atmosphere is extremely lively and international: in [8], Hirzebruch says that he counted on a random day and found 24 nationalities among 79 researchers, with the most frequently represented being Russia, Germany, the USA, France, Japan, Israel, Great Britain, and Iran, in that order. As well as the Arbeitstagung and many long-term activities in various fields, there is an ongoing flow of seminars and talks, many of which are organized together with the university, as well as a cooperation with the Hausdorff Institute for Mathematics, which was founded in 2006 as part of the Hausdorff Center for Mathematics, to which the MPIM also belongs. A further very successful project, also closely coordinated with the university, is the already-mentioned graduate school, which is perhaps the most visible proof that the MPIM does not work in an ivory tower.

16 Hirzebruch and Israel

A seldom mentioned, but always present theme in Hirzebruch's life was his acute awareness of the German crimes under the Nazi regime. One of the rare occasions when these feelings were made explicit was his acceptance speech of the Wolf Prize in the Israeli Knesset in 1988. I was present on that occasion and will never forget the emotions that I felt myself and saw in the audience on hearing the final words of his speech:

> ...As a professor at the University of Bonn, I am one of the successors of the famous mathematicians Felix Hausdorff and Otto Toeplitz. Hausdorff committed suicide in 1942, together with his wife, when deportation to a concentration camp was imminent; Toeplitz emigrated to Israel in 1939 and died there the following year. The memory of these mathematicians is with me always on this trip.

Hirzebruch had an intense relationship with Israel. He first visited the country in 1981, and starting in the late 1980's became actively involved in the organization of mathematical research there. He was asked and agreed to serve in the Advisory Board of the Gelbart Institute of Bar-Ilan University in Ramat-Gan and in the Beirat (advisory body) of the Edmud Landau Research Institute for Mathematical Analysis of the Hebrew University in Jerusalem. He also served in the Evaluation Committee of the mathematics department in the Technion in Haifa and supported the establishment of the Minkowski Center in Tel-Aviv University.

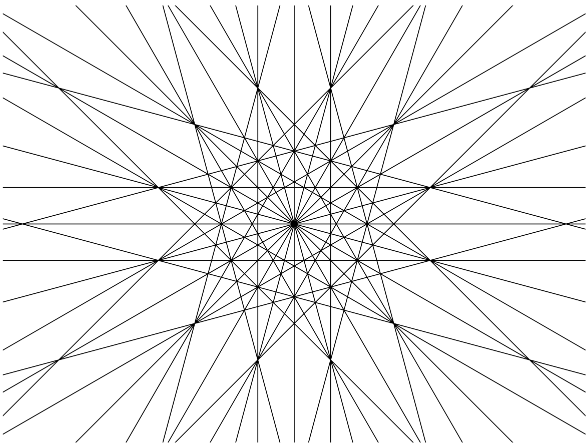
Above all, Hirzebruch had a fundamental role in the Emmy Noether Institute of Bar-Ilan University. In the 1990 meeting of the Advisory Board of the Gelbart Institute (an international visitor program based on a private donation) in Bar-Ilan University, he told the others about the Minerva Foundation, a daughter organization of the Max Planck Society that was founded to foster high-level research and had some 40 centers in Israel in all sciences, each co-funded by the Israeli Institute. At his suggestion and following his advice, the university applied for a joint German-Israeli Minerva center in Mathematics, to be named after the German-Jewish mathematician Emmy Noether, who had fled from Germany to the US in the early 1930's. The application was approved in 1991 and an inauguration ceremony of the Emmy Noether Institute took place in the house of the Israeli Ambassador in Bonn in July 1992. A binational Beirat was appointed by the Bar-Ilan University and the German Ministry of Education and Research, with Hirzebruch being appointed as its chairman. He served in this role for twelve years with great success, helping in particular to extend the network of the institute, to sign exchange agreements with other institutes (including the Banach Institute in Warsaw), to enlarge the visitor program, and to raise the visibility of the center.

The University of Bar-Ilan paid tribute to Hirzebruch's contribution by granting him an honorary doctorate in 2000, and the Emmy Noether Institute organized two major international conferences in algebraic geometry in his honor, "Hirz 65" in 1993 and "Hirz 80" in 2008, on the occasion of his corresponding birthdays. The latter of these was probably the last big conference he attended: he went to all the talks, discussed mathematics and old times with his many friends there, and participated in all the tours (unfortunately falling and breaking a leg during the final one, to the Western Wall caves, but then apologizing profusely for disturbing the agenda!). More about Hirzebruch's relations to Israel can be found in the contribution to [16] by Mina Teicher, director of the Emmy Noether Institute, organizer of the two Bar-Ilan conferences, and Hirzebruch's close friend.

17 "More and More Algebraic Geometry in Combinatorics"

This time the slogan is mine, not Hirzebruch's, but is still very appropriate, since he wrote a number of papers in the 80's and 90's on different links between these two subjects.

Of the links he found, perhaps the most beautiful and unexpected was his application of the theory of algebraic surfaces, and in particular of the Miyaoka-Yau inequality, to the classical problem of arrangements of lines in a plane. This problem, which goes back at least to a paper of Sylvester of 1893, but in fact to a problem about planting trees in an orchard that was posed in the form of a poem in the early 19th century, is very easy to state: one asks what the constraints are for the numbers of meeting points of various valencies in a finite configuration of lines in the plane ("arrangement"). Of course one is interested in cases where the lines meet in many points, of multiplicity as high as possible:



The problem is subtle, because there are arrangements that can be realized combinatorially (e.g., in the projective plane over a finite field) but not over $\mathbb{C}$ and others (like the 12 lines through the 9 inflection points on a plane cubic) that can be realized over $\mathbb{C}$ but not over $\mathbb{R}$. Hirzebruch discovered a beautiful way to apply the Miyaoka-Yau inequality $c_1^2 \leq 3c_2$ for compact complex surfaces to certain ramified coverings of $\mathbb{P}_2(\mathbb{C})$ associated to a complex line arrangement to obtain a result that is stronger than anything that has been obtained by elementary methods. His inequality reads

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{i=5}^{d-2}(2i-9)t_i,$$

where d is the number of lines in the arrangement and t_i (assumed to vanish for $i = d-1$ or $i = d$) is the number of points through which precisely i lines pass. This inequality was used, among other applications, to prove the "bounded negativity" conjecture $(d^2 - c)/n > -4$ (where $n = \sum t_i$ is the number of points and $c = \sum i^2 t_i$ the "self-crossing count") for line arrangements.

In the converse direction, Hirzebruch and his students and coworkers used special line arrangements to construct interesting algebraic surfaces, and in particular surfaces where the Miyaoka-Yau bound is attained. These examples were presented in detail in a book with Gottfried Barthel and Thomas Höfer, and form an important contribution to the "geography problem," still not completely solved, which asks to describe all possible Chern numbers (c_1^2, c_2) of algebraic surfaces. The extreme cases with $c_1^2 = 3c_2$ are particularly interesting, since the Hirzebruch proportionality principle implies that this equality holds for quotients of the complex 2-ball by discrete group actions, and the Miyaoka-Yau theorem states that it holds in no other cases. Hirzebruch was extremely interested in such "ball quotients," which are analogous in many ways to Hilbert modular surfaces, and in particular wrote an 18-page long review, together with Paula Beazley-Cohen, of a book by Deligne and Mostow, in which his examples are linked with those discussed in the book and with the theory of hypergeometric functions of several variables. He continued to occupy himself with this subject for several years and for a long time planned a book on it with Beazley-Cohen, though in the end this joint project was abandoned and work on the book was continued by her alone.

But there were also many smaller projects linking combinatorics and algebraic geometry: a paper relating Euler polynomials and "zigzag permutations" to the holomorphic Euler characteristics of certain algebraic varieties, lectures about the icosahedron (which had also played a role in his investigation of a Hilbert modular surface for $\mathbb{Q}(\sqrt{5})$) and "Fullerenes," and his 2009 Oberwolfach lecture in which Catalan numbers and other numbers of combinatorial interest are related to the Chern numbers of Grassmannians and other Schubert varieties. Brieskorn, writing in [9] about Hirzebruch's later work in algebraic geometry, speaks of "a wealth of beautiful geometry with relations to classical configurations of the 19-th century, but also to modern theoretical physics."

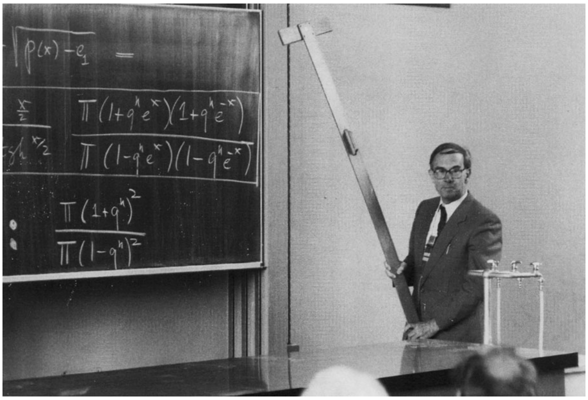


Fig. 5 F. Hirzebruch giving a lecture on elliptic genera at the Arbeitstagung 1987

18 Elliptic Genera

In the late 1980's the theory of elliptic genera was developed by Ochanine, Landweber-Stong, and Witten, and of course Hirzebruch was immediately interested: elliptic genera are the genera, in the sense that he had invented and that we described earlier in this article, having certain classical elliptic functions as their defining power series, so that they attach modular forms to manifolds, thus combining in a deep way two subjects that were very dear to his heart. Moreover, the special values of these modular forms at the two cusps of the modular group in question are precisely the signature and the $\widehat{A}$-genus that he had studied alone and in his joint work with Atiyah. Witten and others had given motivations coming from physics for this theory using the free loop space of a manifold, and this had led Witten to the formulation of a rigidity conjecture for the elliptic genus, later proved by Taubes and by Bott-Taubes, that directly generalized the earlier rigidity theorem of Atiyah and Hirzebruch for spin manifolds with S^1-action. Hirzebruch gave a course on the subject that appeared as a joint book with the students Thomas Berger and Rainer Jung, and also wrote several papers on the subject, including a long joint one "Elliptic genera, involutions, and homogeneous spin manifolds" with Peter Slodowy. Elliptic genera, and some of the results found by Hirzebruch in connection with the course (in particular, a generalization to higher level), have applications in both pure mathematics and mathematical physics, in particular string theory. When asked at the end of his interview with Matthias Kreck [14] whether there were new fields of mathematics that he would like to learn, Hirzebruch answered: "I would not mind to be an expert in string theory"!

19 The Years after Retirement

Hirzebruch retired from the university in 1993 and as director of the MPIM in 1995, each occasion being celebrated by a party and lectures in his honor, in accordance with his dictum that one should never pass up any opportunity to celebrate. There was also a big party in 1992 to which all 52 of his doctoral students were invited and to which 33 came, some from overseas.

During the years following his retirement Hirzebruch remained active both mathematically and in his various national and international roles, and continued to come to the MPI almost every day, when he was not traveling, nearly until his death 19 years later. Much of the mathematics that occupied him during these years was a continuation of earlier directions that have already been described, in particular on relations between combinatorics and algebraic geometry and on the theory of elliptic genera. Here I mention only one other topic, this time from pure number theory (though originally motivated like everything else from topological considerations), which I particularly like because it was joint with me! This was an investigation of "twisted Dedekind sums," which are analogues of the classical Dedekind sums that we had studied in our book, but now involving a quadratic character. Here in his own handwriting is a formula from this work that he wrote down as a present to his friend Taida Hambleton in 1999 on the occasion of a dinner at his beloved restaurant "Maternus" in Bad Godesberg:

p Primzahl $\equiv 1 (4)$ $\quad (d,p)=1$

$$\sum_{k=1}^{p-1} \frac{1}{4\sqrt{p}} \left(\frac{k}{p}\right) \cot\frac{\pi k}{p} \cot\frac{\pi d k}{p}$$

$\in \mathbb{Z}$ für $p>5$

$\in 2\mathbb{Z} \iff \left(\frac{d}{p}\right)=1$

F Hirzebruch 28.6.99

Among other things, we found that the twisted Dedekind sum

$$f(p,d) = \frac{1}{4\sqrt{p}} \sum_{0<k<\sqrt{p}} \left(\frac{k}{p}\right) \cot\frac{\pi k}{p} \cot\frac{\pi k d}{p} \quad \big(p \equiv 1 \ (\mathrm{mod}\ 4) \text{ prime}, p \nmid d\big)$$

appearing on Hirzeburch's card has simple values for small d, e.g.

$$f(p,1) = 2f(p,2) = 3f(p,3) + \frac{h(-3p)}{2} = 4f(p,4) + h(-4p) = 6\zeta_K(-1),$$

where $\zeta_K(-1)$ is the value at $s=1$ of the Dedekind zeta function of $K = \mathbb{Q}(\sqrt{p})$ (and belongs to $\frac{1}{3}\mathbb{Z}$ if $p>5$), and $h(-3p)$ and $h(-4p)$ are class numbers. These formulas are reproduced in Hirzebruch's article [8]. A planned joint paper never got written, but the results will be included in a greatly expanded version of our joint book that is currently in preparation, with Paul Gunnells as a third co-author.

We already discussed some of the international tasks that Hirzebruch continued to perform also after his retirement, such as the chairmanship of the Beirat of the Emmy Noether Institute from 1992 till 2004 and the chairmanship of the Banach Center Council from 1993 till 2001, and of course he also continued to travel and lecture widely. In 1990 he became the first president of the newly formed European Mathematical Society, and served in that role until the end of 1994. Perhaps the high point came in 1998, when he was asked to serve as honorary president of the International Mathematical Congress held in Berlin, the first one to take place in Germany in 94 years, and gave a moving opening address in which he reminded his audience once again of the tragedies of the Nazi era and announced the special activity organized by the German Mathematical Society to honor the memory of the victims.

Many honors had of course come to him even before retirement, but in these later years they multiplied. We have already mentioned the conferences that were held in his honor in Israel and in Poland. His first six honorary doctorates arrived at roughly the same period and at roughly the same frequency as his six grandchildren, so that he would often say jokingly, depending on which side was ahead, that it was time for a new grandchild or a new doctorate—but it was an unequal battle and in the end he received a total of fifteen honorary doctorates, from Warwick, Göttingen, Oxford, Wuppertal, Notre Dame, Dublin, Athens, Potsdam, Konstanz, Berlin, Bar-Ilan, Oslo, Chicago, Bucharest and Augsburg. Then there were the prizes. We have already spoken of the Wolf Prize from Israel, the Lobachevsky prize and Lomonosov Gold Medal from Russia, the Stefan Banach Medal from Poland, and the Seki prize and the Order of the Holy Treasure from Japan, but there were many more, including the Medal of the City of Paris and the Albert Einstein Medal of Switzerland and many from Germany: the Knight Commander's Cross of the Federal Republic of Germany, the Golden Cothenius Medal of the Leopoldina Academy, the Krupp Science Prize, the Helmholtz Medal of the Berlin-Brandenburg Academy of Science, and the Georg Cantor Medal of the Deutsche Mathematiker-Vereinigung. He was also a member or foreign member of more than twenty German and international academies, of which he attached particular importance to the order "Pour le Mérite" because of the extraordinary fellow members that he could get to know at their meetings, almost all of which he attended.

In the last years of his life, Hirzebruch had a series of major and minor health problems, including two serious falls, one at the end of the "HIRZ80" conference in Tel Aviv that we have already mentioned and one only a few weeks before his death. He seldom spoke of these problems and continued, not only to come to his office in the Max Planck Institute nearly every day, but to maintain his interest in both mathematics and mathematicians, making it a point to get to know almost every visitor personally. Five weeks before his death in May 2012, he gave a beautiful two-hour lecture on "The shape of planar algebraic curves defined over the reals" at the MPI that the interested reader can hear on the MPI's on-line "Hirzebruch Collection." His death was reported on by obituaries in numerous newspapers in Germany and abroad. Many people, some coming from far away, came to his funeral to show their affection and pay their respects to a man who had affected all of their lives (Fig. 6).

Half a year before his death, Hirzebruch gave an in-depth interview [14] for the Simons foundation, with his former student Matthias Kreck as the interviewer, that gives a vivid impression of his life and personality. The accompanying article by Joel Segel [15] has the title "Friedrich Hirzebruch, Giant of German Mathematics." There is no better description imaginable.

Fig. 6 F. Hirzebruch's last lecture on "The shape of planar algebraic curves defined over the reals" in May 2012

20 Writings by and about Hirzebruch

Hirzebruch's skills as an expositor of mathematics were just as evident in his writing as in his lectures. It was always beautifully clear, almost entirely free of errors, and written in an informal and very human style that was nevertheless at the same time also extremely precise. Here, for instance are some quite typical sentences from the write-up of his 2009 Oberwolfach lecture about Chern classes, in which, after explaining how to calculate the number of lines on a cubic hypersurface in $\mathbb{P}_3$, a quintic hypersurface in $\mathbb{P}_4$, or a degree 7 hypersurface in $\mathbb{P}_5$, he adds:

> The number of lines on a hypersurface occurred as a Chern number. We had to neglect, for example, the long history of the discovery of the 27 lines on a cubic surface. On my desk there is the classical model of a cubic surface of Clebsch and Klein defined over the reals. Here also the 27 lines are defined over the reals. Thus I can see them any time I wish. In this special case the lines have 10 triple points.

Altogether Hirzebruch published well over a hundred mathematical articles, mostly research but also many expositions of his own work or that of others, including several exposés in the Séminaire Bourbaki. We will not give the full list here, since it is too long and can be found on the Hirzebruch Collection page on the MPIM website (https://hirzebruch.mpim-bonn.mpg.de). We do, however, give a complete list of his many books (which include not only the famous *Topological Methods* but also a number of research monographs or textbooks, often based on courses he gave and sometimes co-authored with people who had attended them), followed by a list of the mathematical articles that were discussed in the main text, together with their numbers in his published *Collected Papers*. In the following and final paragraph "References" we then give a short list, in chronological order, of various more historically-oriented articles written by or about him.

Books by Friedrich Hirzebruch

New Topological Methods in Algebraic Geometry [German, 1956 and 1962; English 1966; Japanese, 1970; Russian, 1973; Chinese 2004. Springer "Classics in Mathematics" 1995]

Sheaf- and Cohomology Theory (with G. Scheja) [German, 1957]

Introduction to the Theory of Vector Bundles and K-Theory (notes by M. Hazewinkel and D. Erle) [1965]
O(n)-Manifolds and Exotic Spheres (with K.H. Mayer) [German, 1968]
Introduction to Functional Analysis (with W. Scharlau) [German, 1971 and 1996]
Differentiable Manifolds and Quadratic Forms (with W.D. Neumann und S. Koh) [1971]
Hilbert Modular Surfaces [1973]
The Atiyah-Singer Theorem and Elementary Number Theory (with D. Zagier) [1974]
Lectures on Hilbert Modular Surfaces (with G. van der Geer) [1981]
Numbers (with H.-D. Ebbinghaus et al.) [German, 1983 and 1992; English, 1996]
Line Configurations and Algebraic Surfaces (with G. Barthel and Th. Höfer) [German, 1987]
Manifolds and Modular Forms (with Th. Berger and R. Jung) [1992 and 1994]

Articles by F. Hirzebruch that are discussed individually in the text:

Note: Numbers like "I.10" or "II.43" refer to papers from Volume I (papers 1–33) or Volume II (papers 34–75) of the *Gesammelte Abhandlungen*. Numbers starting with "III" refer to the informal Volume III (papers 76–93 and a.–g.) that was produced at the MPI in 2001, but since this volume is not published, complete references have been given for these papers. It is planned to prepare and publish a more complete Volume III.

I.1. Über eine Klasse von einfach zusammenhängenden komplexen Mannigfaltigkeiten (1951).
I.2. Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichern (1953).
I.10. Some problems on differentiable and complex manifolds (1954).
I.M2. Neue topologische Methoden in der algebraischen Geometrie (1956).
I.16. Automorphe Formen und der Satz von Riemann-Roch (1958).
I.19/22/26. (with A. Borel) Characteristic classes and homogeneous spaces. I, II, III (1958/59/60).
I.27/32. (with M.F. Atiyah) Vector bundles and homogeneous spaces (1961); (with M.F. Atiyah) Charakteristische Klassen und Anwendungen (1961).
I.21/24. A Riemann-Roch theorem for differentiable manifolds [Bourbaki] (1959); (with M.F. Atiyah) Riemann-Roch theorems for differentiable manifolds (1959).
I.25. (with M.F. Atiyah) Quelques théorèmes de non-plongement pour les variétés différentiables (1959).
I.27. (with M.F. Atiyah) Vector bundles and homogeneous spaces (1961).
I.28. (with M.F. Atiyah) Bott periodicity and the parallelizability of the spheres (1961).
I.32. (with M.F. Atiyah) Charakteristische Klassen und Anwendungen (1962).
II.37. Elliptische Differentialoperatoren auf Mannigfaltigkeiten (1966).
II.38. Über Singularitäten komplexer Flächen (1966).
II.39. Singularities and exotic spheres [Bourbaki] (1967).

II.43. (with M.F. Atiyah) Spin-manifolds and group actions (1970).
II.45. Free involutions on manifolds and some elementary number theory (1971).
II.46. Pontrjagin classes of rational homology manifolds and the signature of some affine hypersurfaces (1971); Pontrjagin classes of rational homology manifold [Notes by S. Morita] (1972).
II.48./51. The Hilbert modular group, resolution of the singularities at the cusps and related problems [Bourbaki] (1971); Hilbert modular surfaces (1973).
II.53. (A. Van de Ven) Hilbert modular surfaces and the classification of algebraic surfaces (1974).
II.60. (with D. Zagier) Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus (1976).
II.61. (with D. Zagier) Classification of Hilbert modular surfaces (1977).
II.69. Arrangements of lines and algebraic surfaces (1983).
II.73. Algebraic surfaces with extreme Chern numbers (report on the thesis of Th. Höfer) (1985).
III.79. (with P. Slodowy) Elliptic genera, involutions, and homogeneous spin manifolds. Geometriae Dedicata **35** (1990) 309–343.
III.82. Mannigfaltigkeiten und Modulformen. Jber. d. DtṀath.-Verein. (1992), 20–38.
III.84. (with P. Beazley-Cohen) Book review: *Commensurabilities among lattices in* $PU(1, n)$, by Pierre Deligne and G. Daniel Mostow. Bull. AMS **32** (1995) 88–105.

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13. Hirzebruch, F.: Why do I like Chern, and why do I like Chern classes? Not. Am. Math. Soc. **58**, 1231–1234 (2011)
14. Kreck, M.: Video interview with Friedrich Hirzebruch. Simons Foundation website (2011)

15. Segel, J.: Friedrich Hirzebruch: Giant of German Mathematics. Science Lives Project. Simons Foundation website (2011)
16. Atiyah, M.F., Zagier, D. (coordinating editors): Friedrich Hirzebruch (1927–2012). Not. Am. Math. Soc. **61**, 2–23 (2014)
17. Atiyah, M.F.: Friedrich Ernst Peter Hirzebruch, 17 October 1927–27 May 2012, Biogr. Mem. Fellows R. Soc. **60**, 229–247 (2014)

Don Zagier is an American mathematician who came to Germany in 1970 to complete his Ph.D. under the supervision of Friedrich Hirzebruch and ended up spending essentially his entire professional life there, though often with parallel positions in other countries (USA, Holland, France, Italy). He began working in topology but then moved to number theory, especially the theory of modular forms, and connections to other fields of mathematics and mathematical physics. His work with Hirzebruch, first on relations between index theory and elementary number theory and then on Hilbert modular surfaces, was decisive throughout his mathematical life.